Construction of Cross Z-Complementary Sequence Set with Large CZC Ratio

Kai Liu, Qing Hai Xu and Xin Meng

July 2, 2024

Construction of Cross Z-complementary Sequence Set with Large CZC Ratio

Kai Liu *, Qinghai Xu, Xin Meng
School of Information Science and Engineering
Yanshan University
Qinhuangdao, China.
liukai@ysu.edu.cn

Abstract

Cross Z-complementary pairs/sequences (CZCPs/CZCSs) are widely used for training sequences in spatial modulation (SM) systems and can achieve superior channel estimation performance in frequency-selective channels, whose aperiodic correlation sums appear as zero correlation zones at both the front-end and back-end offsets of the sequences. Nevertheless, the ZCZ length of the binary CZCP is restricted to half of its length, while the CZCS can result in a larger increase in ZCZ length, and is suitable for SM systems against larger delay expansion. This paper proposes a class of optimal CZCS sets (CZCSSs) with flexible ZCZ length by employing CZCPs and Hadamard products. To improve the parameters of CZCPs, two novel classes of CZCPs are introduced through concatenation construction. The construction results yield new parameters and expand the pool of training sequences available for SM systems.

1 Introduction

Spatial modulation (SM) is a category of MIMO modulation techniques. Multiple transmit antenna (TA) elements are present in an SM system, but only one radio frequency (RF) chain. Within each time slot, the SM symbol can be divided into two parts: one part is called the "spatial symbol", which is responsible for selecting and activating TA elements, and the other part is called the "constellation symbol", which is selected from traditional PSK/QAM constellations and transmitted from active TA elements. "Single RF chain" of SM in principle doesn't permit the transmitter to transmit using the pilots on all TAs simultaneously, so the dense training sequence of conventional MIMO in [12]-[14] is not applicable to SM systems. For this reason, Liu proposed cross Z-complementary

*The authors are supported in part by the Natural Science Foundation of Hebei Province under Grant F2023203066, and in part by the Key Laboratory Project of Hebei Province, China under Grant 202250701010046.

pairs (CZCPs) that can be applied to SM training sequences [4]. On frequency selective channels, the effect of multi-path propagation can be mitigated by the "tail-end zero crosscorrelation zone" and "tail-end zero autocorrelation zone" of the CZCP, while the intersymbol interference between each row can be mitigated by the "front-end zero autocorrelation zone" of the CZCP. The idea of CZCPs is derived from Golay complementary pairs (GCPs) [3] and Z-complementary pairs (ZCPs) [1], which have aperiodic auto-correlation sums (AACSs) for front-end and tail-end ZCZ, as well as aperiodic cross-correlation sums (ACCSs) for tail-end ZCZ. Liu also pointed out that the ZCZ length of CZCP cannot exceed $N/2$, where N is the sequence length. When the ZCZ length reaches half of the sequence length, it is called a perfect CZCP. Cross Z-complementary ratio (CZC_{ratio}) is defined in [5] as the ratio of ZCZ length Z to the maximum possible ZCZ width Z_{max}. When $CZC_{ratio} = 1$, it is referred to as the optimal CZCP. Multiple CZCPs with different CZC_{ratio} are constructed in [4], [5], [7]-[10]. Recently, CZCPs have been expanded to CZCSs[11] and CZCSSs[2].

In the literature, binary quaternary and q-ary CZCPs have been developed. Adhikary used the insertion method to indirectly construct a number of binary CZCPs with larger CZC_{ratio} [5]. He also used Barker codes to construct a class of optimal binary CZCPs and extended the length of binary CZCPs through the Turyn method. Fan proposed several types of binary CZCPs with parameters $(10^{\beta}, 4 \times 10^{\beta-1})$,$(26^{\gamma}, 12 \times 26^{\gamma-1})$,$(10^{\beta}26^{\gamma}, 12 \times 10^{\beta}26^{\gamma-1})$[7], which are also GCPs. Huang used Boolean functions (BFs) to directly construct binary CZCPs, whose $CZC_{ratio} \approx 2/3$[8]. In [9], binary CZCPs of different lengths were constructed using ZCPs and the concatenation method, with the largest CZC_{ratio} being . Zhang searched for the optimal seed CZCP sequence by computer and then constructed binary CZCPs with a larger CZC_{ratio} by combining GCPs and Kronecker products [10]. In [9] and [15], binary CZCPs were mapped to quaternary CZCPs.Liu constructed an optimal q-ary CZCP with a length of $2^m (m \geq 4)$ based on generalized Boolean functions (GBFs) [4], whereas Adhikary constructed a non-optimal q-ary CZCP with a length of $2^{m-1} + 2 (m \geq 4)$ using GBFs [5].To extend the ZCZ length, the concept of CZCS is introduced as the extension of CZCP [4].Kumar directly constructed $(2^{n+1}, 2^{n+1}, 2^{m-1} + 2, 2^{\pi(m-3)} + 1)$-CZCSS using GBFs [2].In this paper, we also propose two methods for constructing CZCPs using concatenation techniques. Based on the literature and our constructed CZCPs, a class of indirect construction methods for CZCSS is proposed, where the set parameters can be optimized

The rest of this paper is organized as follows. In part two, the basic definitions of CZCP and CZCSS are introduced. In part three, two constructions of CZCPs are constructed and CZCSS, and the parameters of constructed results are compared to the literature. A conclusion will then be presented.

2 Basic Concepts

Let **a** and **b** be two complex sequences of length N, some notations are given as follows:

- $\boldsymbol{a} \parallel \boldsymbol{b}$ represents the concatenation of the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$;
- $\overleftarrow{\boldsymbol{a}}$ represents the reverse of $\boldsymbol{a}$;

• $\boldsymbol{a}^*$ represents the complex conjugate of $\boldsymbol{a}$.

Definition 1: Let $\boldsymbol{a} = (a_0, a_1, \cdots, a_{N-1})$ and $\boldsymbol{b} = (b_0, b_1, \cdots, b_{N-1})$ be two sequences of length N, and the aperiodic correlation function of and is defined as

$$\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} \boldsymbol{a}_i \boldsymbol{b}_{i+\tau}^*, 0 \leq \tau \leq N-1, \\ \sum_{i=0}^{N-1-\tau} \boldsymbol{a}_{i-\tau} \boldsymbol{b}_i^*, -(N-1) \leq \tau < 0, \\ 0, |\tau| \geq N. \end{cases} \tag{1}$$

If $\boldsymbol{a} \neq \boldsymbol{b}$, $\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ is called the aperiodic cross-correlation function (ACCF) of $\boldsymbol{a}$ and $\boldsymbol{b}$; if $\boldsymbol{a} = \boldsymbol{b}$, $\rho_{\boldsymbol{a},\boldsymbol{a}}(\tau)$ is called the aperiodic auto-correlation function (AACF) of $\boldsymbol{a}$, represented by $\rho_{\boldsymbol{a}}(\tau)$.

Definition 2[3]: If the AACF sum of sequences $\boldsymbol{a}$ and $\boldsymbol{a}$ of length N satisfies $\rho_{\boldsymbol{a}}(\tau) + \rho_{\boldsymbol{b}}(\tau) = 0$ for $1 \leq \tau \leq N-1$, then $(\boldsymbol{a}, \boldsymbol{b})$ is called GCP.

Definition 3[6]: Let $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ be two GCPs of length N if $\rho_{\boldsymbol{a},\boldsymbol{c}}(\tau) + \rho_{\boldsymbol{b},\boldsymbol{d}}(\tau) = 0$ for $0 \leq \tau \leq N-1$, then $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ are referred to as mate each other.

Definition 4[2]: Given a set $S = \{S^0, S^1, \ldots, S^{K-1}\}$, where each element set S^p is composed of M sequences, namely $S^p = \{s_0^p, s_1^p, \ldots, s_{M-1}^p\}$, $s_l^p = (s_{l,t}^p, 0 \leq t < N)$, where $0 \leq p \leq K-1$, $0 \leq l \leq M-1$. If the set S satisfies the following properties:

$$\begin{aligned} P1 &: \sum_{i=0}^{M-1} \rho(s_i^p)(\tau) = 0,\ |\tau| \in (V_1 \cup V_2) \cap V; \\ P2 &: \sum_{i=0}^{M-1} \rho(s_i^p, s_{i+1}^p)(\tau) = 0, |\tau| \in V_2; \\ P3 &: \sum_{i=0}^{M-1} \rho(s_i^p, s_i^{p'})(\tau) = 0, |\tau| \in \{0\} \cup V_1 \cup V_2; \\ P4 &: \sum_{i=0}^{M-1} \rho(s_i^p, s_{i+1}^{p'})(\tau) = 0, |\tau| \in \cup V_2 \end{aligned} \tag{2}$$

It is called a (K, M, N, Z)-CZCSS, where $s_M^p = s_0^p$, $s_M^{p'} = s_0^{p'}$, $p \neq p'$, $V_1 = \{1, 2, \ldots, Z\}$, $V_2 = \{N-Z, N-Z+1, \ldots, N-1\}$, $V = \{1, 2, \ldots, N-1\}$, $Z \leq N$. If $K = 1$, then S is reduced to a CZCS [11]. If $K = 1$ and $M = 2$, S is then converted to a CZCP.

According to Definition 4, P1 indicates that each CZCP needs to have two zero auto-correlation zones (ZACZs) when considering AACS. They are referred to in this paper as the "front-end ZACZ" and "tail-end ZACZ" with time-shift on V_1 and V_2, respectively. When evaluating ACCS, P2 indicates that each CZCP needs to have a "tail-end zero cross correlation zone (ZCCZ)".

Definition 5[16]: Let $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ and $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ be two CZCPs of length N. If they satisfies the following properties:

$$\begin{aligned} &\rho(\boldsymbol{a}_0, \boldsymbol{a}_1)(\tau) + \rho(\boldsymbol{b}_0, \boldsymbol{b}_1)(\tau) = 0, |\tau| \in V_1 \cup V_2 \cup \{0\}; \\ &\rho(\boldsymbol{a}_1, \boldsymbol{a}_0)(\tau) + \rho(\boldsymbol{b}_1, \boldsymbol{b}_0)(\tau) = 0, |\tau| \in V_1 \cup V_2 \cup \{0\}; \\ &\rho(\boldsymbol{a}_0, \boldsymbol{b}_1)(\tau) + \rho(\boldsymbol{b}_0, \boldsymbol{a}_1)(\tau) = 0, |\tau| \in V_2; \\ &\rho(\boldsymbol{b}_1, \boldsymbol{a}_0)(\tau) + \rho(\boldsymbol{a}_1, \boldsymbol{b}_0)(\tau) = 0, |\tau| \in V_2. \end{aligned} \tag{3}$$

Then $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ and $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ are said to be CZCP mates of each other, where $V_1 = \{1, 2, \ldots, Z\}$, $V_2 = \{N-Z, N-Z+1, \ldots, N-1\}$, $Z \leq N$.

Definition 6[5]: Let $(\boldsymbol{a}, \boldsymbol{b})$ be a CZCP with length N and a ZCZ length of Z. If the maximum achievable length of Z is $Z_{\max}$, then define $CZC_{ratio} = Z/Z_{\max}$. When $CZC_{ratio} = 1$, CZCP is deemed optimal.

When the length of binary CZCPs is $N = 2^{\alpha}10^{\beta}26^{\gamma}$, $Z_{\max}$ is $N/2$, otherwise it is $N/2 - 1$[9].

Lemma 1[17]: For a (K, M, N, Z)-CZCSS $S = \{S^0, S^1, \ldots, S^{K-1}\}$, the upper bound on ZCZ width is given by

$$Z \leq \tfrac{MN}{K} - 1 \tag{4}$$

For the binary CZCSS, we have

$$Z \leq \tfrac{MN}{2K} \tag{5}$$

A q-ary (K, M, N, Z)-CZCSSs is called optimal if $Z = (MN)/K - 1$ for $q > 2$ or $Z = (MN)/2K$ for $q = 2$.

3 The proposed method

Step 1: Let $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ be a (N, Z)-CZCP and $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ be the mate of $(\boldsymbol{a}_0, \boldsymbol{b}_0)$.

Step 2: Set $\alpha = \{x_i\}_{i=0}^{2^n-1}$, $\beta = \{y_i\}_{i=0}^{2^n-1}$, where $x_i = \boldsymbol{a}_0$ or $\boldsymbol{a}_1$, $y_i = \begin{cases} \boldsymbol{b}_0, x_i = \boldsymbol{a}_0 \\ \boldsymbol{b}_1, x_i = \boldsymbol{a}_1 \end{cases}$. $\overline{\alpha} = \{\overline{x_i}\}_{i=0}^{2^n-1}$, where $\overline{x_i} = \begin{cases} \boldsymbol{a}_0, x_i = \boldsymbol{a}_1 \\ \boldsymbol{a}_1, x_i = \boldsymbol{a}_0 \end{cases}$, similarly, $\overline{\beta} = \{\overline{y_i}\}_{i=0}^{2^n-1}$, where $\overline{y_i} = \begin{cases} \boldsymbol{b}_0, x_i = \boldsymbol{a}_1 \\ \boldsymbol{b}_1, x_i = \boldsymbol{a}_0 \end{cases}$ so there are the following equations:

$$\rho_{x_i}(\tau) + \rho_{y_i}(\tau) = 0, |\tau| \in V_1 \cup V_2 \tag{6}$$

$$\rho_{x_i, y_i}(\tau) + \rho_{y_i, x_i}(\tau) = 0, |\tau| \in V_2 \tag{7}$$

$$\rho_{x_i, \overline{x_i}}(\tau) + \rho_{y_i, \overline{y_i}}(\tau) = 0, \forall \tau \tag{8}$$

Step 3: Let $\mathrm{H} = [h_{i,j}]_{2^n \times 2^n}$ be a Hadamard matrix of order $2^n \times 2^n$, so that the matrix S

$$\mathrm{S} = \begin{bmatrix} \mathrm{H} \circ \alpha & \mathrm{H} \circ \beta \\ \mathrm{H} \circ \overline{\alpha} & \mathrm{H} \circ \overline{\beta} \end{bmatrix} \tag{9}$$

where $\circ$ represents Hadamard product. Take the row vector of S to form the sequence set $S = \{S_\mu, 0 \leq \mu < 2^{n+1}\}$.

Theorem 1. *The sequence set S constructed from the above steps is a $(2^{n+1}, 2^{n+1}, N, Z)$-CZCSS.*

Until now, only [2] proposed a class of CZCSS, then the comparison of parameters is shown in Table 1. The direct GBF-based construction proposed in [2],[17] and the indirect construction method proposed in **Theorem1** provide ideas for CZCSS design, despite the

Table 1: Parameter Comparison of CZCSSs

Ref	Sequence Set Parameters	Methods and constraints	Remarks
[2]	$(2^{n+1}, 2^{n+1}, 2^{m-1}+2, 2^{\pi(m-3)}+1)$	GBFs. $m > 4$	Non-optimal
[17]	$(2^k, 2^v, 2^m, 2^{\pi_1(1)-1})$	GBFs. $m, v, k \in \mathbb{Z}^+, v \le k$	when $\pi_1(1) = m-k+v$,optimal
Thm.2	$(2^{n+1}, 2^{n+1}, N, Z)$	Hadamard Matrix and the Hadamard Product of CZCP	when $Z = N/2$, optimal

fact that the two constructions produce non-optimal CZCSSs. **Theorem1** uses Hadamard matrices and the CZCPs with larger CZC_{ratio} to construct the CZCSS with more flexible parameters, and when $Z = N/2$, CZCSS achieves optimal performance. Therefore, the CZCSS derived from **Theorem1** have a higher CZC ratio than that of [2].

Let $(\boldsymbol{a}, \boldsymbol{b})$ be a GCP of length N, then $(\boldsymbol{c}, \boldsymbol{d}) = (\overleftarrow{\boldsymbol{b}}^*, \overleftarrow{-\boldsymbol{a}^*})$ is the mate of $(\boldsymbol{a}, \boldsymbol{b})$. Perform the following two concatenation operations on $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$:

$$ConstructionI \begin{array}{l} \boldsymbol{a}_0 = (\boldsymbol{a}\,\|\boldsymbol{c}\|\,\boldsymbol{a}\,\|\boldsymbol{b}\|\,\boldsymbol{d}\,\|\boldsymbol{b}), \\ \boldsymbol{b}_0 = (\boldsymbol{a}\,\|\boldsymbol{c}\|\,\boldsymbol{a}\,\|{-\boldsymbol{b}}\|-\boldsymbol{d}\,\|{-\boldsymbol{b}}); \end{array} \tag{10}$$

$$ConstructionII \begin{array}{l} \boldsymbol{a}_0 = (\boldsymbol{a}\,\|\boldsymbol{a}\|-\boldsymbol{a}\,\|\boldsymbol{c}\|-\boldsymbol{a}\,\|\boldsymbol{b}\|\,\boldsymbol{b}\,\|{-\boldsymbol{b}}\|\,\boldsymbol{d}\,\|{-\boldsymbol{b}}), \\ \boldsymbol{b}_0 = (\boldsymbol{a}\,\|\boldsymbol{a}\|-\boldsymbol{a}\,\|\boldsymbol{c}\|-\boldsymbol{a}\,\|{-\boldsymbol{b}}\|-\boldsymbol{b}\,\|\boldsymbol{b}\|-\boldsymbol{d}\,\|\boldsymbol{b}). \end{array} \tag{11}$$

Theorem 2. $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ *obtained from the above Construction I is a* $(6N, 2N-1)$*-CZCP,* $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ *obtained from the Construction II is a* $(10N, 3N-1)$*-CZCP.*

The comparison of CZCPs parameters is shown in Table 2. Compared to existing literature, **Theorem 2** uses GCPs and the concatenation operation to obtain CZCPs with larger CZC_{ratio} and new parameter combinations.

4 Proof

Proof of Theorem1. Due to $\rho_{x_i}(\tau) + \rho_{y_i}(\tau) = 0$ for $|\tau| \in V_1 \cup V_2$, $0 \le i < 2^n$, the AACF of S_μ is as follow

$$\rho_{S_\mu}(\tau) = \sum_{i=0}^{2^n-1} h^2_{\mu \bmod 2^n, i}(\rho_{x_i}(\tau) + \rho_{y_i}(\tau)) = 0, |\tau| \in V_1 \cup V_2 \tag{12}$$

Equation (12) satisfies the condition P1 of Definition 4.

$$\begin{array}{l} \rho_{S^i_\mu, S^{i+1}_\mu}(\tau) = \sum_{i=0}^{2^n-2} h_{\mu \bmod 2^n, i} h_{\mu \bmod 2^n, i+1}(\rho_{x_i, x_{i+1}}(\tau) + \rho_{y_i, y_{i+1}}(\tau)) + \\ h_{\mu \bmod 2^n, 2^{n-1}} h_{\mu \bmod 2^n, 0}(\rho_{x_{2^n-1}, y_0}(\tau) + \rho_{y_{2^n-1}, x_0}(\tau)) \end{array} \tag{13}$$

When $x_i = x_{i+1}$, $y_i = y_{i+1}$, obtained from $\rho_{x_i}(\tau) + \rho_{y_i}(\tau) = 0$, $|\tau| \in V_1 \cup V_2$ and $\rho_{x_0,y_0}(\tau) + \rho_{y_0,x_0}(\tau) = 0$, $|\tau| \in V_2$:

$$\rho_{S^i_\mu, S^{i+1}_\mu}(\tau) = 0 \tag{14}$$

When $x_i \neq x_{i+1}$, $y_i \neq y_{i+1}$, obtained from $\rho_{x_i,\overline{x_i}}(\tau) + \rho_{y_i,\overline{y_i}}(\tau) = 0$ for all τ and $\rho_{x_0,\overline{y_0}}(\tau) + \rho_{y_0,\overline{x_0}}(\tau) = 0$ for $|\tau| \in V_2$, so we have

$$\rho_{S^i_\mu, S^{i+1}_\mu}(\tau) = 0 \tag{15}$$

Equations (14) and (15) satisfy the condition P2 of Definition 4.

Let S_e and S_f denote two different rows of S, when $0 \leq e, f < 2^n$ or $2^n \leq e, f < 2^{n+1}$,using the properties of the Hadamard matrix, we have

$$\rho_{S_e,S_f}(\tau) = \sum_{i=0}^{2^n-1} h_{e \bmod 2^n,i} h_{f \bmod 2^n,i} (\rho_{x_i}(\tau) + \rho_{y_i}(\tau)) = 0 \tag{16}$$

Where $|\tau| \in V_1 \cup V_2$.

When $0 \leq e < 2^n$, $2^n \leq f < 2^{n+1}$, $h_{e \bmod 2^n,i} = h_{f \bmod 2^n,i}$, then

$$\rho_{S_e,S_f}(\tau) = \sum_{i=0}^{2^n-1} h^2_{e \bmod 2^n,i} (\rho_{x_i,\overline{x_i}}(\tau) + \rho_{y_i,\overline{y_i}}(\tau)) = 0 \tag{17}$$

Where $0 \leq |\tau| < N$.

Equations (16) and (17) satisfy the condition P3 of Definition 4.

From equation (9), two different rows e, f,

$$\begin{aligned}\rho_{S^i_e, S^{i+1}_f}(\tau) = &\sum_{i=0}^{2^n-2} h_{e \bmod 2^n,i} h_{f \bmod 2^n,i+1} (\rho_{x_i,x_{i+1}}(\tau) + \rho_{y_i,y_{i+1}}(\tau)) + \\ &h_{e \bmod 2^n,2^n-1} h_{f \bmod 2^n,0} \rho_{x_{2^n-1},y_0}(\tau) + h_{e \bmod 2^n,2^n-1} h_{f \bmod 2^n,0} \rho_{y_{2^n-1},x_0}(\tau)\end{aligned} \tag{18}$$

Assuming $x_i = x_{i+1}$, $y_i = y_{i+1}$, then $\rho_{x_i}(\tau) + \rho_{y_i}(\tau) = 0$, $|\tau| \in V_1 \cup V_2$ and $\rho_{x_0,y_0}(\tau) + \rho_{y_0,x_0}(\tau) = 0$, $|\tau| \in V_2$, therefore

$$\rho_{S^i_e, S^{i+1}_f}(\tau) = 0 + \rho_{x_0,y_0}(\tau) + \rho_{y_0,x_0}(\tau) = 0 \tag{19}$$

Assuming $x_i \neq x_{i+1}$, $y_i \neq y_{i+1}$, $\rho_{x_i,\overline{x_i}}(\tau) + \rho_{y_i,\overline{y_i}}(\tau) = 0$ for $\forall\tau$, $\rho_{x_0,\overline{y_0}}(\tau) + \rho_{y_0,\overline{x_0}}(\tau) = 0$, $|\tau| \in V_2$, then

$$\rho_{S^i_e, S^{i+1}_f}(\tau) = 0 + \rho_{x_0,\overline{y_0}}(\tau) + \rho_{y_0,\overline{x_0}}(\tau) = 0 \tag{20}$$

Equations (19) and (20) satisfy the P4 condition of Definition 4. In summary, S is a $(2^{n+1}, 2^{n+1}, N, Z)$-CZCSS.This completes the proof of Theorem1. □

Proof of Theorem2. Firstly, let's prove Construction I.

For $\tau > 0$, according to Definitions 2 and 3, the AACFs of $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ are calculated in the following ways:

From Definition 2 and Definition 3, it can be concluded that:

$\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)=0$, $1\le\tau\le N-1$; $\rho_{\boldsymbol{c}}(\tau)+\rho_{\boldsymbol{d}}(\tau)=0$, $1\le\tau\le N-1$; $\rho^*_{\boldsymbol{a},\boldsymbol{c}}(\tau)+\rho^*_{\boldsymbol{b},\boldsymbol{d}}(\tau)=0$, $0\le\tau\le N-1$.

For $0<\tau\le N-1$, we have

$$\rho_{\boldsymbol{a}_0}(\tau)=2\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{c}}(\tau)+2\rho_{\boldsymbol{b}}(\tau)+\rho_{\boldsymbol{d}}(\tau)+\rho^*_{\boldsymbol{c},\boldsymbol{a}}(N-\tau)+\rho^*_{\boldsymbol{a},\boldsymbol{c}}(N-\tau)+\rho^*_{\boldsymbol{b},\boldsymbol{a}}(N-\tau)+ \rho^*_{\boldsymbol{d},\boldsymbol{b}}(N-\tau)+\rho^*_{\boldsymbol{b},\boldsymbol{d}}(N-\tau) \tag{21}$$

$$\rho_{\boldsymbol{b}_0}(\tau)=2\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{c}}(\tau)+2\rho_{\boldsymbol{b}}(\tau)+\rho_{\boldsymbol{d}}(\tau)+\rho^*_{\boldsymbol{c},\boldsymbol{a}}(N-\tau)+\rho^*_{\boldsymbol{a},\boldsymbol{c}}(N-\tau)-\rho^*_{\boldsymbol{b},\boldsymbol{a}}(N-\tau)+ \rho^*_{\boldsymbol{d},\boldsymbol{b}}(N-\tau)+\rho^*_{\boldsymbol{b},\boldsymbol{d}}(N-\tau) \tag{22}$$

then

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=4\rho_{\boldsymbol{a}}(\tau)+4\rho_{\boldsymbol{b}}(\tau)+2\rho_{\boldsymbol{c}}(\tau)+2\rho_{\boldsymbol{d}}(\tau)+2\rho^*_{\boldsymbol{c},\boldsymbol{a}}(N-\tau)+2\rho^*_{\boldsymbol{a},\boldsymbol{c}}(N-\tau)+ 2\rho^*_{\boldsymbol{b},\boldsymbol{d}}(N-\tau)+2\rho^*_{\boldsymbol{d},\boldsymbol{b}}(N-\tau)=0 \tag{23}$$

Similarly,for $\tau=N$, we have

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=2\rho^*_{\boldsymbol{a},\boldsymbol{c}}(\tau-N)+2\rho^*_{\boldsymbol{c},\boldsymbol{a}}(\tau-N)+2\rho^*_{\boldsymbol{b},\boldsymbol{d}}(\tau-N)+2\rho^*_{\boldsymbol{d},\boldsymbol{b}}(\tau-N)=0 \tag{24}$$

For $N+1\le\tau\le 2N-1$, we have

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=2\rho_{\boldsymbol{a},\boldsymbol{c}}(\tau-N)+2\rho_{\boldsymbol{c},\boldsymbol{a}}(\tau-N)+2\rho_{\boldsymbol{b},\boldsymbol{d}}(\tau-N)+2\rho_{\boldsymbol{d},\boldsymbol{b}}(\tau-N)+ 2\rho^*_{\boldsymbol{a}}(2N-\tau)+2\rho^*_{\boldsymbol{b}}(2N-\tau)=0 \tag{25}$$

For $\tau=2N$, we have

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=2\rho_{\boldsymbol{a}}(\tau-2N)+2\rho_{\boldsymbol{b}}(\tau-2N)=4N \tag{26}$$

For $2N+1\le\tau\le 3N-1$, we have

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=2\rho_{\boldsymbol{a}}(\tau-2N)+2\rho_{\boldsymbol{b}}(\tau-2N)=0 \tag{27}$$

For $3N\le\tau\le 6N-1$, we have

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=0 \tag{28}$$

From the above, it can be obtained that

$$\rho_{\boldsymbol{a}_0}(\tau)+\rho_{\boldsymbol{b}_0}(\tau)=\begin{cases}0, 0<\tau\le 2N-1\\ 4N, \tau=2N\\ 0, 2N+1\le\tau\le 6N-1\end{cases} \tag{29}$$

Similarly, when $\tau < 0$, the conclusion also holds. Therefore, it can be concluded that

$$\rho_{\boldsymbol{a}_0}(\tau) + \rho_{\boldsymbol{b}_0}(\tau) = \begin{cases} 0, 0 < |\tau| \leq 2N-1 \\ 4N, |\tau| = 2N \\ 0, 2N+1 \leq |\tau| \leq 6N-1 \end{cases} \tag{30}$$

The condition C1 of Definitions 4 is satisfied.

Next, the ACCFs of $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ are calculated as follows:

For $4N+1 \leq \tau \leq 6N-1$, we have

$$\rho_{\boldsymbol{a}_0,\boldsymbol{b}_0}(\tau) + \rho_{\boldsymbol{b}_0,\boldsymbol{a}_0}(\tau) = 0 \tag{31}$$

Therefore, when $4N+1 \leq \tau \leq 6N-1$, $\rho_{\boldsymbol{a}_0,\boldsymbol{b}_0}(\tau) + \rho_{\boldsymbol{b}_0,\boldsymbol{a}_0}(\tau) = 0$. Similarly, when $1-6N \leq \tau \leq -1-4N$, $\rho_{\boldsymbol{a}_0,\boldsymbol{b}_0}(\tau) + \rho_{\boldsymbol{b}_0,\boldsymbol{a}_0}(\tau) = 0$. So $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ satisfies the condition C2 of *Definitions* 4 for $4N+1 \leq |\tau| \leq 6N-1$. In summary, $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ obtained from *Construction* I is a $(6N, 2N-1)$-CZCP.

Secondly, let's demonstrate *Construction* II. Similar to *Construction* I it can be concluded that

$$\rho_{\boldsymbol{a}_0}(\tau) + \rho_{\boldsymbol{b}_0}(\tau) = \begin{cases} 0, 1 \leq |\tau| \leq 3N-1 \\ -4N,\ |\tau| = 3N \\ 0, 3N+1 \leq |\tau| \leq 4N-1 \\ -4N,\ |\tau| = 4N \\ 0, 4N+1 \leq |\tau| \leq 10N-1 \end{cases} \tag{32}$$

For $7N+1 \leq |\tau| \leq 10N-1$, we have

$$\rho_{\boldsymbol{a}_0,\boldsymbol{b}_0}(\tau) + \rho_{\boldsymbol{b}_0,\boldsymbol{a}_0}(\tau) = 0 \tag{33}$$

According to (32) and (33), the conditions C1 and C2 of Definitions 4 are satisfied, so $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ is a $(10N, 3N-1)$-CZCP. This completes the proof of Theorem2. □

5 Conclusion

This paper presents a class of optimal CZCSS methods based on CZCPs and their mates, utilizing Hadamard products. Furthermore, to enrich the base sequences, two types of CZCPs are constructed using the concatenation technique and GCPs, thus extending the parameter range of CZCPs. Currently, there are few results on the construction of CZCSSs, with only one type of direct construction method based on GBF proposed in [2]. The CZCSSs constructed in this article can achieve flexible sequence length and ZCZ length. The construction results of this article can provide more options for training sequences in SM systems.

References

[1] P. Fan, W. Yuan, and Y. Tu. Z-complementary binary sequences. *IEEE Signal Processing Letters*, vol. 14, no. 8, pp. 509–512, Aug. 2007

[2] P.Kumar, S.Majhi, S.Paul. A Direct Construction of Cross Z-Complementary Sequence Sets with Large Set Size. *Cryptogr. Commun.* https://doi.org/10.1007/s12095-024-00700-7. Feb. 2024.

[3] M. Golay. Complementary series. *IRE Transactions on Information Theory*, vol. 7, no. 2, pp. 82-87, Apr. 1961.

[4] Z. Liu, P. Yang, Y. L. Guan and P. Xiao. Cross Z-Complementary Pairs for Optimal Training in Spatial Modulation Over Frequency Selective Channels. *IEEE Transactions on Signal Processing*, vol. 68, pp. 1529-1543, Feb. 2020.

[5] A. R. Adhikary, Z. Zhou, Y. Yang and P. Fan. Constructions of Cross Z-Complementary Pairs With New Lengths. *IEEE Transactions on Signal Processing*, vol. 68, pp. 4700-4712, Aug. 2020.

[6] Chin-Chong Tseng and C. Liu. Complementary sets of sequences. *IEEE Transactions on Information Theory*, vol. 18, no. 5, pp. 644-652, Sep. 1972.

[7] C. Fan, D. Zhang, and A. R. Adhikary. New sets of binary cross Z-complementary sequence pairs. *IEEE Communications Letters*, vol. 24, no. 8, pp. 1616–1620, Aug. 2020.

[8] Z. -M. Huang, C. –Y. Pai and C. -Y. Chen. Binary Cross Z-Complementary Pairs With Flexible Lengths From Boolean Functions. *IEEE Communications Letters*, vol. 25, no. 4, pp. 1057-1061, Apr. 2021.

[9] M. Yang, S. Tian, N. Li and A. R. Adhikary. New Sets of Quadriphase Cross Z-Complementary Pairs for Preamble Design in Spatial Modulation. *IEEE Signal Processing Letters*, vol. 28, pp. 1240-1244, May. 2021.

[10] H. Zhang, C. L. Fan, Y. Yang and S. Mesnager. New Binary Cross Z-Complementary Pairs With Large CZC Ratio. *IEEE Transactions on Information Theory*, vol. 69, no. 2, pp. 1328–1336, Feb.2023.

[11] Z. M. Huang, C.Y. Pai, and C. Y. Chen. Cross Z-Complementary Sets for Training Design in Spatial Modulation. *IEEE Transactions on Communications*, vol.70, no. 8, pp. 5030–5045, Aug. 2022.

[12] S. A. Yang and J. Wu. Optimal binary training sequence design for multiple-antenna systems over dispersive fading channels. *IEEE Transactions on Vehicular Technology*, vol. 51, no. 5, pp. 1271-1276, Sep. 2002.

[13] C. Fragouli, N. Al-Dhahir and W. Turin. Training-based channel estimation for multiple-antenna broadband transmissions. *IEEE Transactions on Wireless Communications*, vol. 2, no. 2, pp. 384-391, Mar. 2003.

[14] P. Fan and W. H. Mow. On optimal training sequence design for multiple-antenna systems over dispersive fading channels and its extensions. *IEEE Transactions on Vehicular Technology*, vol. 53, no. 5, pp. 1623-1626, Sep. 2004.

[15] F. Zeng, X. He, Z. Zhang and L. Yan. Quadriphase Cross Z-Complementary Pairs for Pilot Sequence Design in Spatial Modulation Systems. in *IEEE Signal Processing Letters*, vol. 29, pp. 508-512, Jan. 2022.

[16] Z. M. Huang, C. Hu,Z. Liu, C. Y. Chen. On the Mates of Cross Z-Complementary Pairs for Training Sequence Design in Generalized Spatial Modulation. *IEEE Access*, vol. 11, pp. 145231-145237, Dec. 2023.

[17] Z. -M. Huang, C. -Y. Pai, Z. Liu and C. -Y. Chen. Enhanced Cross Z-Complementary Set and Its Application in Generalized Spatial Modulation. 2023.[Online].Available: https://doi.org/10.48550/arXiv.2311.18390.

Table 2: The Comparison of CZCP Parameters

Ref	Parameters	CZC_{ratio}	Methods	Optimality
[4]	$(2N,N)$,$N=2^{\alpha}10^{\beta}26^{\gamma}$	1	GCPs	Yes
	$(2^m,2^{m-1}),m\geq 2$	1	GBF	Yes
[5]	$\left(2^{m-1}+2,2^{\pi(m-3)}+1\right),m\geq 4$	$\leq \frac{1}{2}$	GBF	No
	$(2N+2,N/2+1)$	$\leq \frac{1}{2}$	Insertion	No
	$(12,5)\,(24,11)$	1	Barker code	Yes
	$(12N,5N)\,,(24N,11N)$	$\leq \frac{5}{6},\leq \frac{11}{12}$	Kronecker product and GCPs	No
[7]	$\left(10^{\beta},4\times 10^{\beta-1}\right),\beta\geq 1$	$\frac{4}{5}$	Kronecker product and GCPs	No
	$(26^{\gamma},12\times 26^{\gamma-1}),\gamma\geq 1$	$\frac{12}{13}$		No
	$\left(10^{\beta}26^{\gamma},12\times 10^{\beta}26^{\gamma-1}\right),\gamma\geq 1$	$\frac{12}{13}$		No
[8]	$\left(2^{m-1}+2^{v+1},2^{\pi(v+1)-1}+2^{v}-1\right)$ $m\geq 4,0\leq v\leq m-3$	$\leq \frac{2}{3}$	BF	No
[9]	$(2^{m+2}+2^{m+1},2^{m+1}-1)$	$\leq \frac{2}{3}$	ZCPs and concatenation operation	No
	$(2^{m+4}+2^{m+3}+2^{m+2},2^{m+3}-1)$	$\leq \frac{4}{7}$		No
	$\left(2^{\alpha+2}10^{\beta}26^{\gamma}+4,3\times 2^{\alpha-1}10^{\beta}26^{\gamma}\right)$	$\frac{3}{4}$		No
	$\left(7\times 2^{\alpha+2}10^{\beta}26^{\gamma},3\times 2^{\alpha+2}10^{\beta}26^{\gamma}-1\right)$	$\frac{6}{7}$		No
	$\left(3\times 2^{\alpha+2}10^{\beta}26^{\gamma},5\times 2^{\alpha+1}10^{\beta}26^{\gamma}-1\right)$	$\frac{5}{6}$		No
[10]	$(M,\frac{M}{2}-1),M\in\{6,12,24,28,48,56\}$	1	Computer Search	Yes
	$(MN,(\frac{5M-6}{10})N),N=10^{\beta+1}$	$\frac{5M-6}{5M}$	GCPs and Kronecker product	No
	$MN,(\frac{13M-14}{26})N,N=26^{\gamma+1}$	$\frac{13M-14}{13M}$		No
	$(96,47),(112,55)$	1		Yes
	$(96N,47N)$	$\leq \frac{27}{28}$		No
	$(112N,55N)$	$\leq \frac{55}{56}$		No
[16]	$(2N,2Z)$	$\leq \frac{2Z}{N}$	bit-interleaved	No
Thm.1	$(6N,2N-1)$	$\leq 2/3$	GCPs concatenation operation	No
	$(10N,3N-1)$	$\leq 3/5$		No