Transient Behavior of the M /G / m and M /G/oo Systems

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May 8, 2022

Transient Behavior of the $M/G/m$ and $M/G/\infty$ Systems[1]

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ABSTRACT

It is intended, in this work, to present an approximation process of the $M/G/m$ transient behaviour, *X(t)*, since for it there is not a known expression. It is called diffusion process and the results obtained are in (1-2). The numerical results obtained by Choi and Shin, see (1), are compared with the ones given by the analogous expressions for the $M/G/\infty$ system, see (3). It is concluded that this system can approximate quite well the $M/G/m$ system transient behaviour.

Keywords: Transient behaviour, diffusion processes, $M/G/m$, $M/G/\infty$.

1. INTRODUCTION

Consider a queue system $M/G/m$ at which the customers arrive at instants $t_1, t_2, \ldots$, and the inter-arrivals times $t_{k+1} - t_k, k = 0, 1, 2, \ldots, t_0 = 0$ are independent and identical distributed random variables with exponential distribution with parameter λ_0. The m servers are identical and act in parallel. The service times have a general distribution with mean α and variance σ^2 and are independent from the inter-arrivals times and from the number of the customers in the system. Also $\rho_0 = \lambda_0 \alpha$ and $\rho = \rho_0/m$.

[1] This work was financially supported by FCT through the Strategic Project PEst-OE/EGE/UI0315/2011.

2. APPROXIMATION FOR THE NUMBER OF CUSTOMERS IN THE *M / G / m* SYSTEM

To approximate the number of customers in the *M / G / m* system, Choi and Shin, see (1), considered an elementary return process $\{X(t), t \geq 0\}$ with states space $[0, \infty]$ and with elementary return boundary in *x* = 0. The elementary return process works in the following way:

- When the $X(t)$ path reaches the boundary, it remains there for a random time instant called "suspension time". After the sojourn in the boundary the path jumps for the interior of the region.

In the queues context the "suspension time" in *x* = 0 represents the time interval during which the system is empty. As the arrivals time is Poisson, the "suspension time" has exponential time with parameter λ_0.

Choi and Shin, (1), consider that the return process $\{X(t), t \geq 0\}$ with states space $[0, \infty]$ and $X(0) = x_0$ approximates the customers number in the *M / G / m* system. So, $X(t)$ is specified by the diffusion parameters $a(x)$ and $b(x)$, designated respectively infinitesimal variance and infinitesimal mean, defined as

$$a(x) = \lim_{\Delta t \to 0} \frac{VAR(X(t + \Delta t) - X(t)|\, X(t) = x)}{\Delta t}$$
$$b(x) = \lim_{\Delta t \to 0} \frac{E(X(t + \Delta t) - X(t)|\, X(t) = x)}{\Delta t} \quad (2.1).$$

It is also defined a probability density function $f(x, t|\, x_0)$ by

$$f(x, t|\, x_0)dx = P(x \leq X(t) < x + dx|\, X(0) = x_0) \quad (2.2).$$

Calling $F(x, s|x_0)$ the Laplace-Stieltjes Transform of $f(x, t|\, x_0)$ those authors demonstrated that:

- $k - 1 < x \leq k, k = 1, 2, \dots, m - 1$

$$F_k(x, s|\, x_0) = exp\left(\frac{b_k}{a_k}(x - k)\right)\frac{sh\big(A_k(x - k + 1)\big)}{sh(A_k)}G_k(s|\, x_0) + exp\left(\frac{b_k}{a_k}(x - k + 1)\right)\frac{sh\big(A_k(k - x)\big)}{sh(A_k)}G_{k-1}(s|\, x_0) \quad (2.3),$$

- $m - 1 < x < \infty$

$$F_m(x,s|\,x_0) = exp\left(\left(\frac{b_m}{a_m} - A_m\right)(x-m+1)\right)G_{m-1}(s|\,x_0)$$
$$+\frac{2}{a_mA_m}exp\left(\frac{b_m}{a_m}(x-x_0)\right)\{e^{-A_m(x_0-m+1)}sh(x-m+1)$$
$$-sh\big(A_m(x-x_0)\big)U(x-x_0)1(x_0\geq m)\} \qquad (2.4),$$

where *sh(x)* is the hyperbolic sinus of *x* and the $G_k(s|x_0)$ are given by

$$G_1(s|x_0) = \frac{1}{B_1}(\lambda+s)P(s) - \frac{1}{B_1}1(x_0=0),$$

$$G_2(s|x_0) = \frac{C_2}{B_2}G_1(s) - \frac{1}{B_2}\lambda P(s) - \frac{1}{B_2}1(x_0=1),$$

$$G_k(s|x_0) = \frac{C_k}{B_k}G_{k-1}(s) - \frac{B_{k-1}}{B_k}e^{2\frac{b_{k-1}}{a_{k-1}}}G_{k-2}(s) - \frac{1}{B_k}1(x_0=k-1), k = 3,4,\ldots,m-1,$$

$$G_m(s|\,x_0) = \frac{1}{C_m}B_{m-1}e^{2\frac{b_{m-1}}{a_{m-1}}}G_{m-2}(s) + \frac{1}{C_m}e^{-\left(\frac{b_m}{a_m}+A_m\right)(x_0-m+1)}1(x_0=m-1),$$

where 1(*D*) is the indicator function of *D*, $U(x) = 1(x\geq 0)$ and

$$A_k = \frac{\sqrt{2a_k(s)+b_k^2}}{a_k}, k=1,2,\ldots,m,$$

$$B_k = \frac{a_kA_k}{2}e^{-\frac{b_k}{a_k}}\frac{1}{sh(A_k)}, k=1,2,\ldots,m-1,$$

$$C_k = -\frac{b_{k-1}}{2} + \frac{a_{k-1}A_{k-1}}{2}\frac{ch(A_{k-1})}{sh(A_{k-1})} + \frac{b_k}{2} + \frac{a_kA_k}{2}\frac{ch(A_k)}{sh(A_k)}, k=2,3,\ldots,m-1,$$

$$C_m = -\frac{b_{m-1}}{2} + \frac{a_{m-1}A_{m-1}}{2}\frac{ch(A_{m-1})}{sh(A_{m-1})} + \frac{b_m}{2} + \frac{a_mA_m}{2},$$

being *ch(x)* the hyperbolic cosines of *x* and a_k=*a(k)*and b_k=*b(k)*. Otherwise $P(s) + \int_0^\infty F(x,s|\,x_0)dx = \frac{1}{s}, Re\,(s) > 0.$

An approximation for the stationary distribution of the number of the customers in the system may be obtained making *t* converge to infinite. That is, making $f_k(x) = \lim_{t\to\infty} f_k(x,t|\,x_0)$, $g_k = \lim_{t\to\infty} g_k(t|\,x_0)$, $k = 1,2,\ldots,m-1$, $f_m(x) = \lim_{t\to\infty} f_m(x,t|\,x_0)$, $P = \lim_{t\to\infty} P(t)$ and $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$, since $\rho < 1$

- $0 < x \leq 1$,

$$f_1(x) = \frac{\lambda P}{b_1}\left(e^{2\frac{b_1}{a_1}} - 1\right) \qquad (2.5),$$

- $k-1 < x \leq k, k = 2, 3, \ldots, m-1$,

$$f_k(x) = \frac{\lambda P}{b_1}\left(e^{2\frac{b_1}{a_1}} - 1\right)\left(\prod_{j=2}^{k} e^{2\frac{b_j}{a_j}}\right) e^{2\frac{b_k}{a_k}(x-k)} \qquad (2.6),$$

- $m-1 < x < \infty$,

$$f_m(x) = \frac{\lambda P}{b_1}\left(e^{2\frac{b_1}{a_1}} - 1\right)\left(\prod_{j=2}^{m-1} e^{2\frac{b_j}{a_j}}\right) e^{2\frac{b_m}{a_m}(x-m+1)} \qquad (2.7),$$

where

- When $b_k \neq 0, k = 1, 2, \ldots, m-1$,

$$P = \left(1 - \frac{\lambda}{b_1}\sum_{k=1}^{m-1}\left(\frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+1}}\right) q_k\right)^{-1}$$

- When there is an *i* such that $b_i = 0$ (if it exists it is unique)

$$P = \left(1 - \frac{\lambda}{b_1}\sum_{k=1,k\neq i,i-1}^{m-1}\left(\frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+9}}\right) q_k + q_{i-1}\left(1 + \frac{a_{i-1}}{2b_{i-1}} - \frac{a_i}{2b_{i+1}}\right)\right)^{-1},$$

being $q_1 = \frac{\lambda P}{b_1}\left(e^{2\frac{b_1}{a_1}} - 1\right)$ and $q_k = q_1 \prod_{j=2}^{k} e^{2\frac{b_j}{a_j}}, k = 2, 3, \ldots, m-1$.

3. TESTING THE RESULTS SUPPLIED BY THE DIFFUSION PROCESS

To test the results supplied by $X(t)$ in (1) is presented a comparison between those results and the obtained through simulation considering the systems

- $M/M/3$[2] with $X(0) = 3$,
- $M/H_2/3$ with $X(0) = 0$, being the service probability density function considered $b(x) = 0.3\mu_1 e^{-\mu_1 x} + 0.7\mu_2 e^{-\mu_2 x}$, where $\mu_1 = 5$ and $\mu_2 = \frac{0.7\mu_1}{\rho\mu_1 - 0.3}$ and the following values for ρ:

   a. $\rho = 0.3$ (Weak traffic),

   b. $\rho = 0.5$ (Moderate traffic),

   c. $\rho = 0.7$ (Very intense traffic).

Choi and Shin make $p_{x_0 n}(t) = f(n, t|\, x_0)$ and invert the Laplace-Stieltjes Transform using the method

$$f(t) = \frac{ln2}{t}\sum_{i=1}^{N} V_i f\left(\frac{ln2}{t} i\right)$$

where

$$V_i = (-1)^{\frac{N}{2}} + i \sum_{k=\left[\frac{i+1}{2}\right]}^{min\left(i,\frac{N}{2}\right)} \frac{k^{\frac{N}{2}}(2k)!}{\left(\frac{N}{2} - k\right)!\, k!\, (k-1)!\, (i-k)!\, (2k-i)!},$$

making $N = 10$. The results so obtained are designated by diff.

The simulation results are presented in the form of a 95% Confidence Interval. The interval centre is indicated by sim. And the tails by ci.

To these results were superposed the results obtained for the $M/G/\infty$, designated by inf, in the same circumstances, that are presented in the Tables 1-6:

- $M/M/3$

$$p_{3n}(t) = \sum_{k=0}^{3} \binom{3}{k} \frac{\rho_0^{n-k}}{(n-k)!} e^{-\frac{kt}{\alpha} - \rho_0\left(1 - e^{-\frac{t}{\alpha}}\right)} \left(1 - e^{-\frac{t}{\alpha}}\right)^{n+3-2k} i(n-k),$$

where $i(n-k) = \begin{cases} 1, n \geq k \\ 0, n < k \end{cases}$, see (2),

- $M/H_2/3$

$$p_{0n}(t) = \frac{\left(\lambda \int_0^t [1 - G(v)]dv\right)^n}{n!} e^{-\lambda \int_0^t [1-G(v)]dv}, n = 0, 1, 2, \ldots$$ [3]

with service distribution given above by $b(x)$ and $\rho = \rho_0$, see again (2).

[2] The second M means Exponential Service Time.
[3] λ is the arrivals rate and $G(.)$ is the service time distribution function.

Table 1. System *M* / *M* / 3 (x_0 = 3, λ_0 = 3.0 and ρ = 0.3)

t	Meth.	$p_{30}(t)$	$p_{31}(t)$	$p_{32}(t)$	$p_{33}(t)$	$p_{34}(t)$	$p_{35}(t)$	$p_{36}(t)$	$p_{37}(t)$
0.1	diff	0.0032	0.1791	0.4176	0.3005	0.1160	0.0211	0.0018	0.0001
	inf	0.0176	0.1383	0.3730	0.3758	0.0841	0.0102	0.0008	0.0001
	sim	0.0160	0.1395	0.3685	0.3679	0.0945	0.0122	0.0014	0.0001
	ci	0.0015	0.0066	0.0071	0.0071	0.0055	0.0016	0.0006	0.0002
0.3	diff	0.1276	0.3559	0.2905	0.1455	0.0659	0.0247	0.0074	0.0017
	inf	0.1430	0.3310	0.3105	0.1556	0.0478	0.0102	0.0016	0.0002
	sim	0.1407	0.3270	0.2971	0.1502	0.0605	0.0192	0.0044	0.0007
	ci	0.0048	0.0079	0.0076	0.0055	0.0049	0.0021	0.0008	0.0004
0.5	diff	0.2494	0.3654	0.2310	0.1011	0.0441	0.0182	0.0067	0.0022
	inf	0.2572	0.3674	0.2415	0.0983	0.0282	0.0061	0.0011	0.0002
	sim	0.2485	0.3610	0.2332	0.0983	0.0389	0.0139	0.0043	0.0015
	ci	0.0066	0.0072	0.0056	0.0045	0.0037	0.0024	0.0012	0.0006
0.7	diff	0.3194	0.3632	0.2017	0.0808	0.0333	0.0136	0.0054	0.0020
	inf	0.3267	0.3708	0.2047	0.0736	0.0194	0.0040	0.0007	0.0001
	sim	0.3180	0.3674	0.2011	0.0728	0.0277	0.0101	0.0037	0.0015
	ci	0.0082	0.0062	0.0070	0.0036	0.0018	0.0019	0.0013	0.0006
1.0	diff	0.3689	0.3618	0.1825	0.0674	0.0257	0.0100	0.0039	0.0015
	inf	0.3765	0.3685	0.1769	0.0581	0.0141	0.0027	0.0004	0.0001
	sim	0.3688	0.3650	0.1763	0.0575	0.0216	0.0070	0.0025	0.0005
	ci	0.0074	0.0076	0.0052	0.0045	0.0022	0.0013	0.0006	0.0004
3.0	diff	0.4034	0.3527	0.1699	0.0579	0.0167	0.0067	0.0023	0.0008
	inf	0.4065	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4030	0.3631	0.1628	0.0494	0.0147	0.0049	0.0014	0.0004
	ci	0.0070	0.0088	0.0059	0.0033	0.0022	0.0012	0.0006	0.0003
5.0	diff	0.4033	0.3628	0.1699	0.0579	0.0197	0.0067	0.0023	0.0008
	inf	0.4056	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4049	0.3612	0.1635	0.0505	0.0142	0.0039	0.0013	0.0004
	ci	0.0082	0.0071	0.0062	0.0035	0.0020	0.0011	0.0005	0.0003
7.0	diff	0.4040	0.3631	0.1701	0.0579	0.0197	0.0067	0.0023	0.0008
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4041	0.3642	0.1633	0.0472	0.0147	0.0048	0.0012	0.0003
	ci	0.0077	0.0071	0.0055	0.0041	0.0020	0.0011	0.0005	0.0003
10.0	diff	0.4034	0.3628	0.1699	0.0579	0.0197	0.0067	0.0023	0.0008
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4026	0.3606	0.1647	0.0488	0.0157	0.0053	0.0017	0.0004
	ci	0.0080	0.0089	0.0046	0.0035	0.0022	0.0011	0.0006	0.0004
20.0	diff	0.4037	0.3628	0.1699	0.0579	0.0197	0.0067	0.0023	0.0008
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4092	0.3600	0.1621	0.0471	0.0156	0.0040	0.0012	0.0006
	ci	0.0079	0.0081	0.0048	0.0031	0.0015	0.0007	0.0006	0.0004
∞	diff	0.4034	0.3628	0.1699	0.0579	0.0197	0.0067	0.0023	0.0008
	inf	0.6066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	exact*	0.4035	0.3631	0.1634	0.0490	0.0147	0.0044	0.0013	0.0004

* Stationary value for the *M* / *M* / 3 system.

Table 2. System *M* / *M* / 3 (x_0 = 3, λ_0 = 3.0 and ρ = 0.5)

t	Meth.	$p_{30}(t)$	$p_{31}(t)$	$p_{32}(t)$	$p_{33}(t)$	$p_{34}(t)$	$p_{35}(t)$	$p_{36}(t)$	$p_{37}(t)$
0.1	diff	0.0002	0.0668	0.3691	0.4052	0.1647	0.0223	0.0010	0.0000
	inf	0.0045	0.0627	0.2946	0.4960	0.1242	0.0164	0.0015	0.0001
	sim	0.0050	0.0641	0.2923	0.4877	0.1311	0.0180	0.0017	0.0003
	ci	0.0011	0.0032	0.0068	0.0071	0.0062	0.0024	0.0006	0.0002
0.3	diff	0.0359	0.2303	0.3244	0.2326	0.1297	0.0250	0.0146	0.0029
	inf	0.0467	0.2019	0.3332	0.2657	0.1135	0.0315	0.0064	0.0010
	sim	0.0477	0.1950	0.3220	0.2479	0.1262	0.0449	0.0114	0.0023
	ci.	0.0032	0.0051	0.0063	0.0067	0.0040	0.0031	0.0015	0.0007
0.5	diff	0.0926	0.2722	0.2872	0.1830	0.1051	0.0512	0.0204	0.0066
	inf	0.0979	0.2636	0.3054	0.2043	0.0906	0.0292	0.0073	0.0015
	sim	0.0967	0.2506	0.2924	0.1851	0.1042	0.0470	0.0174	0.0050
	ci	0.0043	0.0071	0.0074	0.0055	0.0034	0.0031	0.0019	0.0012
0.7	diff	0.1348	0.2879	0.2666	0.1592	0.0909	0.0473	0.0217	0.0087
	inf	0.1381	0.2917	0.2859	0.1749	0.0759	0.0251	0.0067	0.0015
	sim	0.1362	0.2807	0.2653	0.1565	0.0886	0.0430	0.0190	0.0074
	ci	0.0060	0.0056	0.0067	0.0058	0.0035	0.0032	0.0023	0.0016
1.0	diff	0.1772	0.2999	0.2514	0.1415	0.0789	0.0422	0.0212	0.0097
	inf	0.1767	0.3122	0.2692	0.1516	0.0628	0.0205	0.0055	0.0012
	sim	0.1666	0.3018	0.2470	0.1377	0.0760	0.0399	0.0167	0.0090
	ci.	0.0049	0.0054	0.0063	0.0058	0.0034	0.0031	0.0019	0.0013
3.0	diff	0.2143	0.3172	0.2391	0.1233	0.0637	0.0330	0.0171	0.0088
	inf	0.2223	0.3343	0.2513	0.1260	0.0474	0.0142	0.0036	0.0008
	sim	0.2095	0.3108	0.2390	0.1185	0.0612	0.0310	0.0150	0.0070
	ci	0.0067	0.0079	0.0068	0.0056	0.0034	0.0031	0.0019	0.0013
5.0	diff	0.2156	0.3182	0.2392	0.1228	0.0631	0.0324	0.0166	0.0086
	inf	0.2231	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2115	0.3140	0.2400	0.1164	0.0577	0.0290	0.0155	0.0068
	ci	0.0066	0.0091	0.0069	0.0058	0.0041	0.0027	0.0022	0.0014
7.0	diff	0.2160	0.3185	0.2394	0.1229	0.0631	0.0324	0.0166	0.0085
	inf	0.2231	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2109	0.3134	0.2380	0.1167	0.0592	0.0297	0.0163	0.0078
	ci	0.0060	0.0065	0.0086	0.0060	0.0035	0.0021	0.0019	0.0015
10.0	diff	0.2157	0.3183	0.2392	0.1228	0.0631	0.0324	0.0166	0.0085
	inf	0.2231	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2074	0.3152	0.2382	0.1162	0.0605	0.0311	0.0160	0.0078
	ci	0.0047	0.0061	0.0069	0.0045	0.0037	0.0030	0.0022	0.0014
20.0	diff	0.2159	0.3183	0.2392	0.1228	0.0631	0.0324	0.0166	0.0085
	inf	0.2231	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2140	0.3162	0.2348	0.1166	0.0591	0.0249	0.0152	0.0079
	ci	0.0059	0.0094	0.0072	0.0045	0.0040	0.0023	0.0019	0.0015
∞	diff	0.2157	0.3183	0.2392	0.1228	0.0630	0.0324	0.0166	0.0085
	inf	0.2231	0.3447	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	exact*	0.2105	0.3158	0.2368	0.1148	0.0592	0.0296	0.0148	0.0074

* Stationary value for the *M* / *M* / 3 system.

Table 3. System *M / M* / 3 (x_0 = 3, λ_0 = 3.0 and ρ = 0.7)

t	Meth.	$p_{30}(t)$	$p_{31}(t)$	$p_{32}(t)$	$p_{33}(t)$	$p_{34}(t)$	$p_{35}(t)$	$p_{36}(t)$	$p_{37}(t)$
0.1	diff	0.0000	0.0306	0.3094	0.4650	0.1951	0.0210	0.0005	0.0000
	inf	0.0018	0.0353	0.2367	0.5574	0.1467	0.0201	0.0019	0.0001
	sim	0.0019	0.0358	0.2328	0.5523	0.1526	0.0218	0.0025	0.0003
	ci	0.0007	0.0033	0.0066	0.0073	0.0059	0.0021	0.0008	0.0003
0.3	diff	0.0143	0.1493	0.3028	0.2780	0.1777	0.0739	0.0196	0.0034
	inf	0.0203	0.1291	0.3025	0.3211	0.1622	0.0510	0.0114	0.0020
	sim	0.0195	0.1263	0.2930	0.2937	0.1760	0.0656	0.0175	0.0042
	ci	0.0027	0.0057	0.0069	0.0049	0.0058	0.0038	0.0022	0.0011
0.5	diff	0.0433	0.1884	0.2758	0.2253	0.1534	0.0826	0.0342	0.0109
	inf	0.0455	0.1798	0.2923	0.2595	0.1446	0.0566	0.0170	0.0040
	sim	0.0452	0.1712	0.2768	0.2297	0.1519	0.0806	0.0313	0.0101
	ci	0.0034	0.0073	0.0063	0.0066	0.0055	0.0039	0.0026	0.0014
0.7	diff	0.0672	0.2033	0.2593	0.2000	0.1385	0.0825	0.0412	0.0171
	inf	0.0670	0.2058	0.2823	0.2327	0.1317	0.0556	0.0185	0.0050
	sim	0.0652	0.1922	0.2595	0.2043	0.1375	0.0812	0.0371	0.0152
	ci	0.0038	0.0071	0.0077	0.0051	0.0053	0.0049	0.0028	0.0020
1.0	diff	0.0894	0.2130	0.2458	0.1809	0.1256	0.0798	0.0454	0.0229
	inf	0.0890	0.2264	0.2745	0.2190	0.1195	0.0521	0.0184	0.0055
	sim	0.0810	0.2102	0.2448	0.1822	0.1251	0.0798	0.0424	0.0197
	ci	0.0037	0.0059	0.0070	0.0040	0.0054	0.0050	0.0032	0.0017
3.0	diff	0.1104	0.2166	0.2260	0.1570	0.1079	0.0731	0.0486	0.0316
	inf	0.1209	0.2555	0.2699	0.1900	0.1003	0.0424	0.0149	0.0045
	sim	0.1026	0.2122	0.2229	0.1568	0.1073	0.0714	0.0486	0.0324
	ci	0.0049	0.0064	0.0071	0.0074	0.0042	0.0044	0.0035	0.0021
5.0	diff	0.1075	0.2111	0.2105	0.1537	0.1065	0.0733	0.0500	0.0337
	inf	0.1224	0.2571	0.2700	0.1891	0.0993	0.0417	0.0146	0.0044
	sim	0.1018	0.2097	0.2171	0.1530	0.1034	0.0712	0.0478	0.0333
	ci	0.0046	0.0066	0.0067	0.0049	0.0040	0.0043	0.0033	0.0027
7.0	diff	0.1057	0.2079	0.2176	0.1522	0.1061	0.0736	0.0508	0.0349
	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044
	sim	0.0987	0.2079	0.2121	0.1508	0.1049	0.0744	0.0479	0.0328
	ci	0.0042	0.0075	0.0064	0.0061	0.0046	0.0041	0.0040	0.0030
10.0	diff	0.1040	0.2050	0.2150	0.1507	0.1055	0.0737	0.0513	0.0356
	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044
	sim	0.0956	0.2029	0.2127	0.1536	0.1001	0.0721	0.0522	0.0349
	ci	0.0048	0.0056	0.0061	0.0061	0.0048	0.0048	0.0032	0.0027
20.0	diff	0.1028	0.2026	0.2126	0.1494	0.1049	0.0736	0.0517	0.0363
	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044
	sim	0.0920	0.2032	0.2161	0.1465	0.1027	0.0725	0.0497	0.0348
	ci	0.0036	0.0064	0.0055	0.0050	0.0056	0.0045	0.0040	0.0027
∞	diff	0.1024	0.2012	0.2122	0.1491	0.1048	0.0736	0.0517	0.0363
	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044
	exact*	0.0957	0.2010	0.2110	0.1477	0.1034	0.0724	0.0507	0.0355

* Stationary value for the *M / M* / 3 system.

Table 4. System $M / H_2 / 3$ ($x_0 = 0$, $\lambda_0 = 3.0$ and $\rho = 0.3$)

t	Meth.	$p_{30}(t)$	$p_{31}(t)$	$p_{32}(t)$	$p_{33}(t)$	$p_{34}(t)$	$p_{35}(t)$	$p_{36}(t)$	$p_{37}(t)$
0.1	diff	0.7627	0.2416	0.0356	0.0033	0.0002	0.0000	0.0000	0.0000
	inf	0.7765	0.1964	0.0249	0.0021	0.0001	0.0000	0.0000	0.0000
	sim	0.7778	0.1958	0.0239	0.0022	0.0002	0.0000	0.0000	0.0000
	ci	0.0043	0.0041	0.0025	0.0005	0.0002	0.0000	0.0000	0.0000
0.3	diff	0.5596	0.3241	0.1090	0.0264	0.0055	0.0009	0.0001	0.0000
	inf	0.5714	0.3196	0.0895	0.0167	0.0023	0.0003	0.0000	0.0000
	sim	0.5698	0.3171	0.0930	0.0172	0.0025	0.0004	0.0000	0.0000
	ci	0.0048	0.0051	0.0036	0.0016	0.0005	0.0004	0.0001	0.0000
0.5	diff	0.4822	0.3429	0.1413	0.0428	0.0119	0.0030	0.0007	0.0001
	inf	0.4879	0.3502	0.1257	0.0301	0.0054	0.0008	0.0001	0.0000
	sim	0.4847	0.3500	0.1271	0.0300	0.0069	0.0009	0.0003	0.0000
	ci	0.0061	0.0064	0.0048	0.0022	0.0009	0.0004	0.0002	0.0001
0.7	diff	0.4441	0.3640	0.1556	0.0516	0.0163	0.0049	0.0014	0.0004
	inf	0.4488	0.3596	0.1440	0.0385	0.0077	0.0012	0.0002	0.0000
	sim	0.4462	0.3565	0.1460	0.0395	0.0089	0.0022	0.0004	0.0002
	ci	0.0064	0.0068	0.0043	0.0023	0.0012	0.0007	0.0003	0.0002
1.0	diff	0.4208	0.3482	0.1650	0.0580	0.0199	0.0066	0.0021	0.0007
	inf	0.4332	0.3639	0.1564	0.0448	0.0096	0.0017	0.0002	0.0000
	sim	0.4235	0.3618	0.1536	0.0444	0.0119	0.0039	0.0007	0.0001
	ci	0.0066	0.0084	0.0030	0.0025	0.0011	0.0009	0.0003	0.0001
3.0	diff	0.4046	0.3471	0.1703	0.0625	0.0229	0.0084	0.0031	0.0011
	inf	0.4066	0.3659	0.1646	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4006	0.3694	0.1642	0.0479	0.0161	0.0043	0.0012	0.0005
	ci	0.0073	0.0062	0.0033	0.0031	0.0016	0.0007	0.0004	0.0003
5.0	diff	0.4035	0.3459	0.1699	0.0623	0.0228	0.0084	0.0031	0.0011
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4061	0.3647	0.1595	0.0481	0.0151	0.0044	0.0015	0.0004
	ci	0.0059	0.0051	0.0043	0.0019	0.0018	0.0009	0.0005	0.0003
7.0	diff	0.4064	0.3468	0.1710	0.0627	0.0230	0.0084	0.0031	0.0011
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4047	0.3628	0.1623	0.0481	0.0157	0.0045	0.0010	0.0007
	ci	0.0064	0.0065	0.0046	0.0026	0.0016	0.0007	0.0004	0.0003
10.0	diff	0.4024	0.3450	0.1694	0.0621	0.0228	0.0084	0.0031	0.0011
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4033	0.3646	0.1631	0.0488	0.0141	0.0045	0.0014	0.0003
	ci	0.0082	0.0064	0.0049	0.0035	0.0013	0.0008	0.0004	0.0002
20.0	diff	0.4057	0.3482	0.1708	0.0626	0.0230	0.0084	0.0031	0.0011
	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000
	sim	0.4045	0.3652	0.1610	0.0478	0.1151	0.0045	0.0014	0.0005
	ci	0.0060	0.0050	0.0040	0.0024	0.0019	0.0006	0.0004	0.0002
∞	inf	0.4066	0.3659	0.1647	0.0494	0.0111	0.0020	0.0003	0.0000

Table 5. System $M / H_2 / 3$ ($x_0 = 0$, $\lambda_0 = 3.0$ and $\rho = 0.5$)

t	Meth.	$p_{30}(t)$	$p_{31}(t)$	$p_{32}(t)$	$p_{33}(t)$	$p_{34}(t)$	$p_{35}(t)$	$p_{36}(t)$	$p_{37}(t)$
0.1	diff	0.7537	0.2579	0.0405	0.0036	0.0001	0.0000	0.0000	0.0000
	inf	0.7672	0.2032	0.0269	0.0024	0.0002	0.0000	0.0000	0.0000
	sim	0.7691	0.2019	0.0263	0.0026	0.0002	0.0000	0.0000	0.0000
	ci	0.0040	0.0039	0.0016	0.0006	0.0001	0.0000	0.0000	0.0000
0.3	diff	0.5104	0.3350	0.1373	0.0390	0.0087	0.0015	0.0002	0.0000
	inf	0.5269	0.3376	0.1082	0.0231	0.0037	0.0005	0.0001	0.0000
	sim	0.5273	0.3321	0.1127	0.0233	0.0038	0.0007	0.0001	0.0000
	ci	0.0052	0.0066	0.0043	0.0015	0.0006	0.0004	0.0001	0.0000
0.5	diff.	0.3997	0.3408	0.1844	0.0711	0.0238	0.0069	0.0017	0.0003
	diff	0.4109	0.3655	0.1625	0.0482	0.0107	0.0019	0.0003	0.0000
	inf	0.4123	0.3590	0.1642	0.0486	0.0125	0.0029	0.0004	0.0001
	sim	0.0052	0.0555	0.0045	0.0029	0.0010	0.0006	0.0003	0.0001
	ci								
0.7	diff	0.3371	0.3315	0.2060	0.0917	0.0370	0.0134	0.0044	0.0013
	inf	0.3461	0.3672	0.1948	0.0689	0.0183	0.0039	0.0007	0.0001
	sim	0.3486	0.3625	0.1934	0.0662	0.0213	0.0061	0.0015	0.0003
	ci	0.0046	0.0060	0.0041	0.0032	0.0016	0.0007	0.0004	0.0002
1.0	diff	0.2883	0.3192	0.2196	0.1089	0.0506	0.0219	0.0089	0.0033
	inf	0.2923	0.3595	0.2211	0.0906	0.0279	0.0069	0.0014	0.0002
	sim	0.2960	0.3511	0.2199	0.0847	0.0323	0.0106	0.0040	0.0009
	ci	0.0055	0.0057	0.0049	0.0041	0.0019	0.0015	0.0005	0.0004
3.0	diff	0.2257	0.2867	0.2228	0.1266	0.0713	0.0397	0.0218	0.0118
	inf	0.2256	0.3359	0.2501	0.1241	0.0462	0.0138	0.0034	0.0007
	sim	0.2162	0.3218	0.2415	0.1102	0.0564	0.0269	0.0139	0.0070
	ci	0.0048	0.0059	0.0048	0.0043	0.0018	0.0017	0.0014	0.0012
5.0	diff	0.2199	0.2813	0.2205	0.1126	0.0726	0.0415	0.0236	0.0134
	inf	0.2232	0.3347	0.2510	0.1254	0.0470	0.0141	0.0035	0.0008
	sim	0.2166	0.3150	0.2357	0.1141	0.0577	0.0288	0.0145	0.0086
	ci	0.0064	0.0056	0.0058	0.0041	0.0017	0.0022	0.0014	0.0015
7.0	diff	0.2205	0.2824	0.2213	0.1272	0.0731	0.0420	0.0241	0.0138
	inf	0.2131	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2109	0.3192	0.2292	0.1166	0.0593	0.0305	0.0154	0.0058
	ci	0.0048	0.0066	0.0048	0.0041	0.0024	0.0019	0.0014	0.0015
10.0	diff	0.2182	0.2797	0.2196	0.1264	0.0727	0.0418	0.0241	0.0138
	inf	0.2231	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2113	0.3142	0.2355	0.1127	0.0595	0.0305	0.0162	0.0093
	ci	0.0035	0.0047	0.0038	0.0036	0.0032	0.0021	0.0022	0.0011
20.0	diff	0.2197	0.2817	0.2203	0.1270	0.0731	0.0420	0.0242	0.0139
	inf	0.2131	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008
	sim	0.2122	0.3122	0.2358	0.1143	0.0580	0.0307	0.0165	0.0088
	ci	0.0046	0.0044	0.0050	0.0048	0.0032	0.0017	0.0014	0.0011
∞	inf	0.2231	0.3347	0.2510	0.1255	0.0471	0.0141	0.0035	0.0008

Table 6. System $M / H_2 / 3$ (x_0 = 0, λ_0 = 3.0 and ρ = 0.7)

t	Meth.	$p_{30}(t)$	$p_{31}(t)$	$p_{32}(t)$	$p_{33}(t)$	$p_{34}(t)$	$p_{35}(t)$	$p_{36}(t)$	$p_{37}(t)$
0.1	diff	0.7499	0.2690	0.0416	0.0028	0.0001	0.0000	0.0000	0.0000
	inf	0.7635	0.2060	0.0278	0.0025	0.0002	0.0000	0.0000	0.0000
	sim	0.7654	0.2054	0.0262	0.0027	0.0002	0.0000	0.0000	0.0000
	ci	0.0045	0.0044	0.0019	0.0007	0.0002	0.0000	0.0000	0.0000
0.3	diff	0.4878	0.3399	0.1515	0.0450	0.0099	0.0016	0.0002	0.0000
	inf	0.5082	0.3440	0.1164	0.0263	0.0044	0.0006	0.0001	0.0000
	sim	0.5071	0.3421	0.1186	0.0257	0.0057	0.0008	0.0001	0.0000
	ci	0.0047	0.0065	0.0039	0.0021	0.0013	0.0004	0.0001	0.0000
0.5	diff	0.3616	0.3340	0.2046	0.0874	0.0311	0.0092	0.0022	0.0004
	inf	0.3776	0.3678	0.1791	0.0581	0.0142	0.0028	0.0004	0.0001
	sim	0.3769	0.3677	0.1773	0.0579	0.0156	0.0036	0.0008	0.0003
	ci	0.0056	0.0078	0.0048	0.0033	0.0014	0.0010	0.0003	0.0002
0.7	diff	0.2874	0.3130	0.2267	0.1154	0.0514	0.0200	0.0068	0.0020
	inf	0.3007	0.3613	0.2171	0.0870	0.0261	0.0063	0.0013	0.0006
	sim	0.2990	0.3632	0.2146	0.0818	0.0293	0.0090	0.0024	0.0006
	ci	0.0040	0.0052	0.0049	0.0027	0.0016	0.0014	0.0007	0.0004
1.0	diff	0.2262	0.2862	0.2366	0.1384	0.0736	0.0356	0.0156	0.0062
	inf	0.2332	0.3395	0.2471	0.1199	0.0436	0.0127	0.0031	0.0006
	sim	0.2359	0.3370	0.2408	0.1118	0.0465	0.0182	0.0065	0.0023
	ci	0.0055	0.0076	0.0054	0.0042	0.0028	0.0014	0.0011	0.0005
3.0	diff	0.1327	0.2103	0.2113	0.1525	0.1073	0.0735	0.0489	0.0315
	inf	0.1316	0.2663	0.2709	0.1829	0.0927	0.0376	0.0127	0.0037
	sim	0.1223	0.2496	0.2404	0.1508	0.0313	0.0610	0.0363	0.0206
	ci	0.0030	0.0057	0.0042	0.0038	0.0037	0.0024	0.0018	0.0013
5.0	diff	0.1177	0.1908	0.1964	0.1459	0.1072	0.0778	0.0557	0.0393
	inf	0.1235	0.2582	0.2701	0.1883	0.0985	0.0412	0.0144	0.0043
	sim	0.1073	0.2261	0.2309	0.1383	0.0969	0.0664	0.0455	0.0322
	ci	0.0035	0.0043	0.0034	0.0049	0.0031	0.0029	0.0024	0.0021
7.0	diff	0.1132	0.1847	0.1908	0.1428	0.1063	0.0786	0.0577	0.0420
	inf	0.1226	0.2573	0.2701	0.1889	0.0991	0.0416	0.0146	0.0044
	sim	0.1039	0.2160	0.2183	0.1440	0.0936	0.0662	0.0502	0.0337
	ci	0.0034	0.0044	0.0038	0.0043	0.0025	0.0030	0.0020	0.0022
10.0	diff	0.1083	0.1777	0.1847	0.1190	0.1044	0.0781	0.0583	0.0433
	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044
	sim	0.0994	0.2060	0.2125	0.1374	0.0958	0.0730	0.0485	0.0371
	ci	0.0033	0.0058	0.0048	0.0048	0.0038	0.0033	0.0027	0.0025
20.0	diff	0.1060	0.1743	0.1813	0.1368	0.1032	0.0778	0.0586	0.0442
	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044
	sim	0.0970	0.2034	0.2079	0.1336	0.0938	0.0687	0.0503	0.0369
	ci.	0.0027	0.0046	0.0039	0.0034	0.0036	0.0036	0.0031	0.0026
∞	inf	0.1225	0.2572	0.2700	0.1890	0.0992	0.0417	0.0146	0.0044

4. CONCLUSIONS

If the results supplied by $X(t)$ are quite good the given by the $M/G/\infty$ system are not inferior. And often they are much better, that is closer to the interval centre of the simulation results.

The approximations supplied by the $M/G/\infty$ system is particularly good when, simultaneously ρ, t and n are little. And, beyond any doubt the expressions for $p_{x_0 n}(t)$ in the case of the $M/G/\infty$ system is much simpler than the ones of the $X(t)$ process.

Indeed, the results supplied by the $M/G/\infty$ system are as good as or better than the ones of the diffusion process, demanding much fewer complex computations.

REFERENCES

1. B. D. Choi and N. W. Shin. Transient Diffusion Approximation for M/G/m System. Journal of Operations Research Society of Japan, 34(3), 306-328 (1991).

2. M. A. M. Ferreira. Comportamento Transeunte e Período de Ocupação e Sistemas de Filas de Espera sem Espera. PhD thesis presented at ISCTE, Lisboa (1995).

3. M. A. M. Ferreira and M. Andrade. The Ties Between the $M/G/\infty$ Queue System Transient Behaviour and the Busy Period. International Journal of Academic Research, 1(1), 84-92, (2009).

4. M. A. M. Ferreira. Redes de filas de espera. Master thesis presented at IST – UTL, Lisboa (1987).

5. M. Andrade. A Note on Foundations of Probability. Journal of Mathematics and Technology, 1 (1), 96-98, 2010.

6. M. A. M. Ferreira and M. Andrade. M/M/m/m Queue System Transient Behavior. Journal of Mathematics and Technology, Vol. 1 (1), 49-65, 2010.

7. M. A. M. Ferreira and M. Andrade. Fundaments of Theory of Queues. International Journal of Academic Research, 3 (1), part II, 427-429, 2011.

8. M. A. M. Ferreira and M. Andrade. An Infinite Servers Nodes Network in the Study of a Pensions Fund. International Journal of Latest Trends in Finance and Economic Sciences, 1 (2), 91-94, 2011.

9. M. A. M. Ferreira and M. Andrade. Queue Networks Models with More General Arrival Rates. International Journal of Academic Research, 4 (1), PART A, 5-11, 2012.

10. M. A. M. Ferreira and J. A. Filipe. Solving Logistics Problems using M/G/∞ Queue Systems Busy Period. Aplimat- Journal of Applied Mathematics, 3 (3), 207-212, 2010.

11. M. A. M. Ferreira and J. A. Filipe. Economic Crisis: Using M/G/oo Queue System Busy Period to Solve Logistics Problems in an Organization. *China-USA Business* Review, 9 (9), 59-63, 2010.

12. M. A. M. Ferreira, M. Andrade, J. A. Filipe and M. Coelho. Statistical Queuing Theory with Some Applications. International Journal of Latest Trends in Finance and Economic Sciences, 1 (4), 190-195, 2011.

13. J. Figueira and M. A. M. Ferreira. Representation of a Pensions Fund by a Stochastic Network with Two Nodes: an Exercise. Portuguese Review of Financial Markets, 2 (1), 75-81, 1999.