On Robins Inequality for Positive Integers and Related Bounds

Emmanuel Elima

December 11, 2022

ON ROBINS INEQUALITY FOR POSITIVE INTEGERS AND RELATED BOUNDS

EMMANUEL ELIMA

Abstract. Let n be a positive integer. We use known estimates of some arithmetic functions to derive lower bounds of n for which Robin's inequality holds.

1. Introduction and Results

Let n be a positive integer with sum of divisor function $\sigma(n) := \sum_{d|n} d$. Robins inequality is the inequality

$$\sigma(n) < e^{\gamma} n \log\log n, \tag{1.1}$$

where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. Robin [1] proved that the Riemann Hypothesis is true if and only if inequality (1.1) holds for all $n > 5041$. Inequality (1.1) is known to hold for a few families of integers but the general case still remains an open problem. In the case of t-free integers, Choie et al. [2] proved that if n does not satisfy (1.1), then it must be even, neither square free nor square full and divisible by a fifth power of a prime. Their result has ever since been improved with Axler [3] recently proving that (1.1) holds for every 21-free integer n. As a consequence, the following equivalence of the Riemann hypothesis was formulated.

Proposition 1.1. [See Corollary 2.5 in [3].]
The Riemann hypothesis is true if and only if Robin's inequality (1.1) *holds for every* 21*-full integer* n.

We contribute a partial result to Proposition 1.1 by proving a new family of 21-full integers for which inequality (1.1) holds as stated below.

Theorem 1.2. *Let n be a t-full integer with k distinct prime divisors. Inequality* (1.1) *holds for all* $t > 0.44k\log(k\log k) + 0.49k + 1$.

For the case of bounds, Ramanujan proved that Robin's inequality holds for all sufficiently large values of n. We refine this result by proving the following lower bound for n for which inequality (1.1) holds.

Theorem 1.3. *Let n be a positive integer wuth k distinct prime divisors. Then inequality* (1.1) *holds for all n satisfying* $\log p_k\Big(1 + \dfrac{1}{\log^2 p_k}\Big) < \log\log n$*, where p_k is the k^{th} prime.*

In the case of distinct prime divisors, we prove the following trivial lower bound for k.

Lemma 1.4. *Let n be a positive integer with k distinct prime divisors. Then inequality* (1.1) *holds for all* $k \leq 12$.

Proof. We consider n to be 21-full since Proposition 1.1 implies inequality (1.1) holds for all 21-free integers. If n is 21-full, then there exists a prime divisor q of n such that q^{21} divides n. Thus $n \geq 2^{21} \prod_{i=2}^{k} p_i$. Calculations show that the inequality

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{k} \frac{p_i}{p_i - 1} < \log\log(2^{21} \prod_{i=2}^{k} p_i) \leq \log\log n \tag{1.2}$$

holds for all $k \leq 12$, where the first inequality in (1.2) follows from inequality (2.2).

□

As a consequence of Theorem 1.3 and Lemma 1.4, we prove the following explicit upper bound for integers that do not satisfy Robin's inequality.

Theorem 1.5. *Let n be a positive integer with k distinct prime divisors. If n does not satisfy inequality* (1.1), *then* $n \leq (k \log k)^{1.31k}$.

2. Proof of Theorem 1.3

We can write $n = \prod_{i=1}^{k} q_i^{\alpha_i}$, where q_i are distinct primes and $\alpha_i \in \mathbb{Z}^+$. We notice that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{k} \frac{q_i}{q_i - 1}\Big(1 - \frac{1}{q_i^{\alpha_i+1}}\Big) < \prod_{i=1}^{k} \frac{q_i}{q_i - 1}. \tag{2.1}$$

Since the sequence $\left\{\frac{p_i}{p_i - 1}\right\}$ over the primes is strictly decreasing , we have $\prod_{i=1}^{k} \frac{q_i}{q_i - 1} < \prod_{i=1}^{k} \frac{p_i}{p_i - 1}$. Thus (2.1) becomes

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{k} \frac{p_i}{p_i - 1} < e^{\gamma} \log p_k \Big(1 + \frac{1}{\log^2 p_k}\Big), \tag{2.2}$$

where the last inequality in (2.2) follows from Corollary 1 in [4]. By hypothesis, we have

$$e^{\gamma}\Big(1 + \frac{1}{\log^2 p_k}\Big) < e^{\gamma} \log\log n \tag{2.3}$$

Combining (2.2) and (2.3) completes the proof.

3. Proof of Theorem 1.5

Proof. We proceed by proving that inequality (1.1) holds for all $n > (k \log k)^{1.31k}$. Suppose $n > (k \log k)^{1.31k}$. The case $k \leq 12$ trivially follows from Lemma 1.4. For the case $k > 12$, we prove that

$$\log p_k \Big(1 + \frac{1}{\log^2 p_k}\Big) < \log\log n, \tag{3.1}$$

from which Theorem 1.3 implies inequality (1.1). By taking exponent on both sides, inequality (3.1) is equivalent to

$$p_k \exp\left(\frac{1}{\log p_k}\right) < \log n. \tag{3.2}$$

We have $p_k \geq p_{13} = 41$, from which it follows that $\exp\left(\frac{1}{\log p_k}\right) < 1.31$.

Inequality (3.2) becomes

$$p_k \exp\left(\frac{1}{\log p_k}\right) < 1.3 p_k < 1.31 k \log(k \log k) < \log n. \tag{3.3}$$

Where the second inequality in (3.3) follows from the fact that $p_k < k \log(k \log k)$ (See equation 3.13 in [4]) and the last inequality in (3.3) follows by hypothesis.
We have proved that inequality (3.1) holds for all $n > (k \log k)^{1.31k}$, hence by Theorem 1.3, inequality (1.1) must hold. □

4. Proof of Theorem 1.2

Proof. We consider the case $k > 12$ since the case $k \leq 12$ follows from Lemma 1.4.

Suppose n is a t-full integer, then $n \geq 2^{t-1} \prod_{i=1}^{k} p_i$, where p_i is the i^{th} prime.

Let $\vartheta(p_k) = \sum_{i=1}^{k} \log p_i$. We have

$$\log n \geq (t-1)\log 2 + \vartheta(p_k) > (t-1)\log 2 + k \log(k \log k) - k, \tag{4.1}$$

where the last inequality in (4.1) follows from $\vartheta(p_k) > k \log(k \log k) - k$. (See Proposition 5.1 in [5]).
From (3.3), we have

$$p_k \exp\left(\frac{1}{\log p_k}\right) < 1.31 k \log(k \log k). \tag{4.2}$$

If the inequality

$$1.31 k \log(k \log k) < (t-1)\log 2 + k \log(k \log k) - k \tag{4.3}$$

holds, then it follows from (4.1) and (4.2) that $\log p_k \left(1 + \frac{1}{\log^2 p_k}\right) < \log \log n$ from which Theorem 1.3 implies that inequality (1.1) holds. But inequality (4.3) can be written as $t > 0.44 k \log(k \log k) + 0.49k + 1$ which then concludes the proof. □

References

[1] Robin, G. (1984). *Grandes valeurs de la fonction somme des diviseurs et hypothese de Riemann.* J. Math. pures appl, **63**(2), 187-213.

[2] Choie, Y., Lichiardopol, N., Moree, P., & Sol, P. (2007). *On Robins criterion for the Riemann hypothesis.* Journal de thorie des nombres de Bordeaux, **19**(2), 357-372.

[3] Axler, C. (2021). *On Robin's inequality.* arXiv preprint arXiv:2110.13478.

[4] Rosser, J. B., & Schoenfeld, L. (1962). *Approximate formulas for some functions of prime numbers.* Illinois Journal of Mathematics, **6**(1), 64-94.

[5] Axler, C. New estimates for the n-th prime number, preprint, 2017. Available at arxiv. org/1706.03651.

Department of Mathematics, University of Saint Joseph Mbarara, p.o.box, 218 Mbarara Uganda

E-mail address: elima@usj.ac.ug