Short Note on the Riemann Hypothesis

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Abstract

Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Robin inequality. We know that the Robin inequality is true for all $n > 5040$ which are not divisible by 2. In addition, we prove the Robin inequality is true for all $n > 5040$ which are divisible by 2. In this way, we show the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

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1. Results

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [1].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n.

Theorem 1.2. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *that are not divisible by* 2 *[2].*

In addition, we know that:

Theorem 1.3. $\mathsf{Robins}(n)$ *holds for all* $10^{10^{10}} \geq n > 5040$ *[3].*

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Let $h(n)$ be defined as

$$h(n) = \prod_{q \mid n} \frac{q}{q-1}.$$

These are known results:

Theorem 1.4. *[2]. For* $n > 1$*:*

$$f(n) < h(n).$$

Theorem 1.5. *[4]. For* $n \geq 3$*:*

$$h(n) < e^{\gamma} \times \log\log n + \frac{2.50637}{\log\log n}.$$

Let's prove our main result:

Theorem 1.6. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *that are divisible by* 2.

Proof. Let's assume that $n > 5040$ is divisible by 2. We have that

$$f(n) \leq f(2) \times f(\frac{n}{2})$$

since the function $f(n)$ is submultiplicative (that is $f(q \times r) \leq f(q) \times (r)$) [2]. We use that theorem 1.4 to show that

$$f(2) \times f(\frac{n}{2}) \leq f(2) \times h(\frac{n}{2}) = \frac{f(2)}{h(2)} \times h(n) = \frac{3}{4} \times h(n)$$

since $f(2) = \frac{3}{2}$ and $h(2) = 2$. According to the theorem 1.5, we obtain that

$$f(n) \leq \frac{3}{4} \times h(n) < \frac{3}{4} \times \left(e^{\gamma} \times \log\log n + \frac{2.50637}{\log\log n}\right).$$

Hence, it is enough to prove that

$$\frac{3}{4} \times \left(e^{\gamma} \times \log\log n + \frac{2.50637}{\log\log n}\right) \leq e^{\gamma} \times \log\log n$$

which is equivalent to

$$\frac{3}{4} \times \left(1 + \frac{2.50637}{e^{\gamma} \times (\log\log n)^2}\right) \leq 1$$

after of dividing the both sides of the inequality by $e^{\gamma} \times \log\log n$. We know that $\mathsf{Robins}(n)$ holds for all $10^{10^{10}} \geq n > 5040$ due to the theorem 1.3. Consequently, we would have that

$$\left(\frac{3}{4} + \frac{3}{4} \times \frac{2.50637}{e^{\gamma} \times (\log\log n)^2}\right) < \left(\frac{3}{4} + \frac{3}{4} \times \frac{2.50637}{e^{\gamma} \times (\log\log 10^{10^{10}})^2}\right)$$

for $n > 10^{10^{10}}$. In this way, it is enough to show that

$$\left(\frac{3}{4} + \frac{3}{4} \times \frac{2.50637}{e^{\gamma} \times (\log\log 10^{10^{10}})^2}\right) \leq 1$$

which is the same as

$$\frac{3}{4} \times \frac{2.50637}{e^{\gamma} \times (\log \log 10^{10^{10}})^2} \leq \frac{1}{4}$$

that is equal to

$$\frac{3 \times 2.50637}{e^{\gamma} \times (\log \log 10^{10^{10}})^2} \leq 1$$

after of multiplying by 4. Finally, we need to prove that

$$3 \times 2.50637 \leq e^{\gamma} \times (\log \log 10^{10^{10}})^2$$

which is trivially true and therefore, the proof is complete. □

This result implies the following consequences:

Theorem 1.7. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$.

Proof. This is a direct consequence of theorems 1.2 and 1.6 □

Theorem 1.8. *The Riemann Hypothesis is true*

Proof. This is true because of the theorems 1.1 and 1.7. □

References

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