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Robin's Criterion on Superabundant Numbers

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Abstract

A trustworthy proof for the Riemann hypothesis has been considered as the Holy Grail of Mathematics by several authors. The Riemann hypothesis is the assertion that all non-trivial zeros of the Riemann zeta function have real part $\frac{1}{2}$. Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\log$ is the natural logarithm. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. If the Riemann hypothesis is false, then there are infinitely many superabundant numbers n such that the Robin's inequality is unsatisfied. In this note, we show that the Robin's inequality always holds for large enough superabundant numbers. By reductio ad absurdum, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Robin's inequality, Superabundant numbers, Sum-of-divisors function, Prime numbers

MSC Classification: 11M26 , 11A41 , 11A25

1 Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$. We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The Ramanujan's Theorem states that if the Riemann hypothesis is true, then the previous inequality holds for large enough n. Next, we have the Robin's Theorem:

Proposition 1 $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [1, Theorem 1 pp. 188].*

It is known that $\mathsf{Robin}(n)$ holds for many classes of natural numbers n.

Proposition 2 $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *such that* $q \leq e^{31.018189471}$*, where* q *is the largest prime factor of* n *[2, Theorem 4.2 pp. 4].*

Superabundant numbers were defined by Leonidas Alaoglu and Paul Erdős (1944). In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [3, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following properties for the superabundant numbers:

Proposition 3 *If* n *is superabundant, then* n *is a Hardy-Ramanujan integer [4, Theorem 1 pp. 450].*

Proposition 4 *[4, Theorem 7 pp. 454]. Let* n *be a superabundant number such that* p *is the largest prime factor of* n*, then*

$$p \sim \log n, \quad (n \to \infty).$$

Proposition 5 *[4, Theorem 9 pp. 454]. For some constant $c > 0$, the number of superabundant numbers less than x exceeds*

$$\frac{c \cdot \log x \cdot \log \log x}{(\log \log \log x)^2}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \ \ for \ \ (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

Proposition 6 *Every colossally abundant number is superabundant [4, pp. 455].*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Proposition 7 *If the Riemann hypothesis is false, then there are infinitely many colossally abundant numbers $n > 5040$ such that* $\mathsf{Robin}(n)$ *fails (i.e.* $\mathsf{Robin}(n)$ *does not hold) [1, Proposition pp. 204].*

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Proposition 8 $\mathsf{Robin}(n)$ *holds for all natural numbers $n > 5040$ such that $\nu_2(n) \leq 19$ [5, Theorem 1 pp. 2].*

Proposition 9 *[4, Lemma 1 pp. 451]. Let n be a superabundant number such that q is a prime factor of n, then*

$$q^{\nu_q(n)} < 2^{\nu_2(n)+2}.$$

Proposition 10 *[4, pp. 454]. Let n be a large enough superabundant number such that p is the largest prime factor of n and $2 \leq q \leq p$, then*

$$q^{\nu_q(n)} < 2 \cdot p \cdot \log p.$$

Proposition 11 *[4, Theorem 5 pp. 452]. Let n be a superabundant number such that $\nu_q(n) = k$, p is the largest prime factor of n, $2 \leq q \leq p$ and $q < (\log p)^{\alpha}$, where*

α is a constant, then

$$\log \frac{q^{k+2}-1}{q^{k+2}-q} < \frac{\log q}{p \cdot \log p} \cdot \left\{1 + O\left(\frac{(\log\log p)^2}{\log p \cdot \log q}\right)\right\}.$$

Putting all together yields the proof of the Riemann hypothesis.

2 Central Lemma

The following is a key Lemma.

Lemma 12 *If the Riemann hypothesis is false, then there are infinitely many superabundant numbers n such that* $\mathsf{Robin}(n)$ *fails.*

Proof This is a direct consequence of Propositions 1, 6 and 7. □

3 Main Insight

This is the main insight.

Lemma 13 *Let $n = \prod_{i=1}^{k} q_i^{a_i}$ be a large enough superabundant number such that $q_k > e^{31.018189471}$ is the largest prime factor of n, then*

$$q_k^2 \geq q_i^{a_i}$$

for every $1 \leq i \leq k$.

Proof Suppose that

$$q_k^2 < q_i^{a_i}$$

for some prime q_i such that $\nu_{q_i}(n) = a_i$. By Proposition 9, we have

$$q_k^2 < q_i^{a_i} < 2^{a_1+2}.$$

That's the same as

$$2 < \frac{(a_1+2)\cdot\log(2)}{\log q_k}$$

after of applying the logarithm by transitivity. We know that

$$(a_1+2)\cdot\log(2) < \log(8 \cdot q_k \cdot \log q_k)$$

by Proposition 10 since n is large enough. So,

$$\frac{(a_1+2)\cdot\log(2)}{\log q_k} < 1 + \frac{\log(8 \cdot \log q_k)}{\log q_k}.$$

However, we have

$$1 + \frac{\log(8 \cdot \log q_k)}{\log q_k} < 2$$

for $q_k > e^{31.018189471}$. In this way, we obtain a contradiction and thus, the proof is complete by reductio ad absurdum. □

4 Main Corollary

This is the principal Corollary.

Corollary 14 *Let n be a large enough superabundant number such that $p > 3$ is the largest prime factor of n, then*

$$p < 2^{\nu_2(n)-1}$$

and

$$p < 3^{\nu_3(n)-1}.$$

Proof Let $q \in \{2, 3\}$ and $\nu_q(n) = k$. For every large enough superabundant number n, there is a constant α such that $q < (\log p)^\alpha$. For example, we can take $\alpha = 2.5$ since $(\log p)^{2.5} \geq (\log 5)^{2.5} > 3$. We will use the following inequality

$$\frac{t}{t+1} < \log(1+t), \quad (t > 0).$$

From the previous inequality, we notice that

$$\begin{aligned}
\log \frac{q^{k+2}-1}{q^{k+2}-q} &= \log\left(1 + \frac{q-1}{q^{k+2}-q}\right) \\
&> \frac{\frac{q-1}{q^{k+2}-q}}{\frac{q-1}{q^{k+2}-q}+1} \\
&= \frac{q-1}{(q^{k+2}-q)\cdot(\frac{q-1}{q^{k+2}-q}+1)} \\
&= \frac{q-1}{(q-1)+(q^{k+2}-q)} \\
&= \frac{q-1}{q^{k+2}-1} \\
&> \frac{1}{3\cdot q^{k+1}}.
\end{aligned}$$

Hence, there is a constant $C > 0$ such that

$$q^k > C \cdot \frac{p \cdot \log p}{\log q}$$

by Proposition 11. Putting $c = \frac{C}{\log q}$, then we obtain that

$$c \cdot p \cdot \log p < q^k,$$

where c is a positive constant. We deduce that

$$c \cdot \log p > 3$$

by Proposition 4 for large enough n. Therefore, the proof is done. □

5 Main Theorem

This is the main theorem.

Theorem 15 *The Riemann hypothesis is true.*

Proof There are infinitely many superabundant numbers by Proposition 5. Let $n > 5040$ be a large enough superabundant number. Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of this superabundant number n as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents, since n must be a Hardy-Ramanujan integer by Proposition 3. Suppose that $\mathsf{Robin}(n)$ fails. So,

$$f(n) \geq e^{\gamma} \cdot \log \log n.$$

We know that

$$\begin{aligned} f(n) &= f(2^{\nu_2(n)}) \cdot f(\frac{n}{2^{\nu_2(n)}}) \\ &= \left(2 - \frac{1}{2^{\nu_2(n)}}\right) \cdot f(\frac{n}{2^{\nu_2(n)}}) \\ &< 2 \cdot f(\frac{n}{2^{\nu_2(n)}}) \\ &= f(2 \cdot 3) \cdot f(\frac{n}{2^{\nu_2(n)}}) \\ &\leq f(2 \cdot 3) \cdot f\left(\frac{n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) \cdot f\left(\frac{3^{\nu_3(n)}}{q_k^2}\right) \\ &= f\left(\frac{2 \cdot 3 \cdot n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) \cdot f\left(\frac{3^{\nu_3(n)}}{q_k^2}\right) \end{aligned}$$

since $f(\ldots)$ is multiplicative and submultiplicative [3, pp. 369]. We have

$$f\left(\frac{2 \cdot 3 \cdot n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) < e^{\gamma} \cdot \log \log \left(\frac{2 \cdot 3 \cdot n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right)$$

by Proposition 8. Therefore, we obtain that

$$f\left(\frac{3^{\nu_3(n)}}{q_k^2}\right) \cdot e^{\gamma} \cdot \log \log \left(\frac{2 \cdot 3 \cdot n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) > e^{\gamma} \cdot \log \log n$$

which is the same as

$$f\left(\frac{3^{\nu_3(n)}}{q_k^2}\right) \cdot \log \log \left(\frac{2 \cdot 3 \cdot n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) > \log \log n.$$

However, we know that

$$2^{\nu_2(n)} > 2 \cdot q_k$$

and

$$3^{\nu_3(n)} > 3 \cdot q_k$$

by Corollary 14, because of n is large enough. Consequently, we can see that necessarily,

$$\left(\frac{2 \cdot 3 \cdot n \cdot q_k^2}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) < n.$$

Moreover, we have

$$1 \geq f\left(\frac{3^{\nu_3(n)}}{q_k^2}\right)$$

for the value of $0 < \frac{3^{\nu_3(n)}}{q_k^2} \leq 1$ by Proposition 2 and Lemma 13 since n is large enough. In this way, we obtain a contradiction under the assumption that $\mathsf{Robin}(n)$ fails. To sum up, the study of this arbitrary large enough superabundant number n reveals that $\mathsf{Robin}(n)$ holds on anyway. Accordingly, $\mathsf{Robin}(n)$ holds for all large enough superabundant numbers n. This contradicts the fact that there are infinitely many superabundant numbers n, such that $\mathsf{Robin}(n)$ fails when the Riemann hypothesis is false according to Lemma 12. By reductio ad absurdum, we prove that the Riemann hypothesis is true. □

6 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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