Finite Difference Method for Perona-Malik Model with Fractional Derivative and Its Application in Image Processing

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***Abstract*—In this work, we consider the Perona-Malik (PM) model with fractional derivatives and its application for image processing. This model is obtained from the standard PM equation by replacing the ordinary derivative with a fractional derivative. the numerical resolution of this model is based on the finite difference method, we analyse efficient numerical methods for the fractional model, and we give practical experiments with natural images which are showing that the fractional approach is more efficient than the ordinary integer one. the proposed model has good performance in visual quality and high signal to noise ratio (SNR)/ Peak signal to noise (PSNR)**

***Keywords*- Perona-Malik; image processing; fractional derivative; Finite difference**

I. Introduction

In recent years, the use of fractional partial differential models has become increasingly popular and has also been widely applied various applications in different fields of research [20], including electromagnetism [12], stochastics, fractals, diffusion processes [17], complex networks, image processing ...etc. different models using fractional partial differential equations have been proposed, and there has been significant interest in developing numerical schemes for their solution [3], [5] such as finite element/difference methods or other methods.

In this context several authors have used models with fractional derivatives in image processing, the more studied model was proposed by Perona and Malik, researchers have proposed models based on (PM) by adding or modifying one or more parameters , in [4] the authors propose to modify the classical PM model by introducing the Caputo-Fabrizio fractional gradient inside the diffusivity function, in [18] the authors reinterpret the Perona–Malik model in the language of Gaussian scale mixtures and derive some extensions of the model, in [19] they developed a new noise removal model by combining the modified isotropic diffusion model and the modified Perona-Malik (PM) model, in [14], [15] new diffusion coefficients are proposed for Image denoising

In this work we study a numerical approach to the Perona-Malik (PM) model with Caputo's time-fractional derivatives and its application for image processing

A. Perona-Malik model with time-fractional derivative

Let $\Omega \subset \mathbb{R}^2$ denote a bounded rectangular domain of $\mathbb{R}^2$ We consider the following Perona-Malik problem :

$$\begin{cases} {}^{c}D_t^{\alpha} u(\mathbf{x},t) = div\big(G(\|\nabla u(\mathbf{x},t)\|)\nabla u(\mathbf{x},t)\big) & \text{in } \Omega \times I, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times I, \\ u(\mathbf{x}, t=0) = u_0(\mathbf{x}) & \text{on } \Omega. \end{cases} \tag{1}$$

where :

- The operator ${}^{c}D_t^{\alpha}$ is the time fractional derivative of order α in Caputo sense with $0 < \alpha < 1$ and
$$ {}^{c}D_t^{\alpha} u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x},t)}{\partial s} \frac{ds}{(t-s)^{\alpha}} $$
- $u(\mathbf{x},t)$ represents the smoothed image intensity function in the position $\mathbf{x} = (x,y) \in \Omega$ in time t
- $u_0(\mathbf{x})$ is the original image intensity function (noisy image)
- ∇ (respectively. $div(.)$) denotes the gradient (respectively divergence) operator
- $\dfrac{\partial u}{\partial n}$ is the normal derivative to the boundary
- $I = [0,T]$ where $T > 0$
- Γ designates the Gamma function
- G the diffusion coefficient

Pietro Perona and Jitendra Malik proposed in 1990 two functions as diffusion coefficient:

$$ G(\|\nabla u\|) = e^{-\left(\frac{\|\nabla u\|}{\lambda}\right)^2} $$

and

$$ G(\|\nabla u\|) = \frac{1}{1 + \left(\dfrac{\|\nabla u\|}{\lambda}\right)^2} $$

where :

λ is a positive constant controls the sensitivity to edges, is usually fixed manually(methodologies to estimate the contrast parameter λ is given in [13] and [16]) or as a function of the noise in the image.
Other expressions for the diffusion coefficient are given in [14] and [15]

II. Numerical Schemes

A. Discretization in time: a finite difference scheme

Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$ an uniform discretization of $[0,T]$ of step Δt where $\Delta t = T/n$ ($t_k = k\Delta t$, $k = 0,1,...,n$)
We discretize the derivative operator in Caputo's sense by a finite difference approach by:
for all $0 \le k \le n-1$:

$$\begin{aligned} {}^cD_t^\alpha u(\mathbf{x},t_{k+1}) &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\frac{\partial u(\mathbf{x},s)}{\partial s}\frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(\mathbf{x},t_{j+1})-u(\mathbf{x},t_j)}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{ds}{(t_{k+1}-s)^\alpha}+\overline{R}_{k+1} \end{aligned} \tag{2}$$

where $\overline{R}_{k+1}$ is the truncation error satisfying :

$$\overline{R}_{k+1} \le C_u \Delta t^{2-\alpha}$$

for a positive constant c_u depend only on u (see [5])
by computing $\displaystyle\int_{t_j}^{t_{j+1}}\frac{ds}{(t_{k+1}-s)^\alpha}$, we write (2) as:

$$\begin{aligned} {}^cD_t^\alpha u(\mathbf{x},t_{k+1}) &= \frac{1}{(\Delta t)^\alpha\Gamma(2-\alpha)}\sum_{j=0}^{k}((k-j+1)^{1-\alpha}-(k-j)^{1-\alpha}) \\ &\quad (u(\mathbf{x},t_{j+1})-u(\mathbf{x},t_j))+\overline{R_{k+1}} \end{aligned} \tag{3}$$

or by a change of index as :

$$\begin{aligned} {}^cD_t^\alpha u(\mathbf{x},t_{k+1}) &= \frac{1}{(\Delta t)^\alpha\Gamma(2-\alpha)}\sum_{j=0}^{k}((j+1)^{1-\alpha}-(j)^{1-\alpha}) \\ &\quad (u(\mathbf{x},t_{k-j+1})-u(\mathbf{x},t_{k-j}))+\overline{R_{k+1}} \end{aligned} \tag{4}$$

let us take $c_j = (1+j)^{1-\alpha} - j^{1-\alpha}$ for all $j = 0,1...,k$
and introduce the parameter $\alpha_0 = \Gamma(2-\alpha)\Delta t^\alpha$
we can easily verify that :

$$\sum_{j=0}^{k}(c_j - c_{j+1}) + c_{k+1} = (1-c_1) + \sum_{j=1}^{k-1}(c_j - c_{j+1}) + c_k$$

And note that $c_0 = 1$
we can write 4 as

$$\begin{aligned} {}^cD_t^\alpha u(\mathbf{x},t_{k+1}) &= \frac{1}{\alpha_0}\big(u(\mathbf{x},t_{k+1})-(1-c_1)u(\mathbf{x},t_k) \\ &\quad -\sum_{j=1}^{k-1}B_j u(\mathbf{x},t_{k-j}) - c_k u(\mathbf{x},t_0)\big)+\overline{R}_{k+1} \end{aligned} \tag{5}$$

where $B_j = c_j - c_{j+1}$
Let be $u_k(\mathbf{x}) \approx u(\mathbf{x},t_k)$ the approximation of $u(\mathbf{x},t_k)$
we write finally :

$${}^cD_t^\alpha u^{k+1} \simeq \frac{1}{\alpha_0}\big(u^{k+1}-(1-c_1)u^k-\sum_{j=1}^{k-1}B_j u^{k-j}-c_k u^0\big) \tag{6}$$

B. Discretization in space

Let be Δx and Δy be the space steps such that $\Delta x = \Delta y = 1$
Let us take

$$A(u(\mathbf{x},t)) := div\big(G(\|\nabla u(\mathbf{x},t)\|)\nabla u(\mathbf{x},t)\big) \tag{7}$$

we can write

$$A(u(\mathbf{x},t)) = \frac{\partial}{\partial x}\left(G(\|\nabla u\|)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(G(\|\nabla u\|)\frac{\partial u}{\partial y}\right) \tag{8}$$

we pose :

$$\phi(\mathbf{x},t) = G(\|\nabla u(\mathbf{x},t)\|)\frac{\partial u(\mathbf{x},t)}{\partial x}$$

and

$$\psi(\mathbf{x},t) = G(\|\nabla u(\mathbf{x},t)\|)\frac{\partial u(\mathbf{x},t)}{\partial y}$$

we've using a centered finite difference

$$\frac{\partial\phi(x_i,y_j,t_k)}{\partial x} = \frac{\phi(x_{i+1/2},y_j,t_k)-\phi(x_{i-1/2},y_j,t_k)}{\Delta x}+O(\Delta x) \tag{9}$$

$$\simeq G(|\nabla u^k_{i+1/2,j}|)(u^k_{i+1,j}-u^k_{i,j}) - G(|\nabla u^k_{i-1/2,j}|)(u^k_{i-1,j}-u^k_{i,j}) \tag{10}$$

where $u^k_{i,j}$ is the approximation of $u(x_i,y_j,t_k)$
and

$$G(|\nabla u^n_{i+1/2,j}|) = G(|u^n_{i+1,j}-u^n_{i,j}|)$$

and

$$G(|\nabla u^n_{i-1/2,j}|) = G(|u^n_{i,j}-u^n_{i-1,j}|)$$

the same for

$$\frac{\partial\phi(x_i,y_j,t_k)}{\partial y} = \frac{\phi(x_i,y_{j+1/2},t_k)-\phi(x_{i-1/2},y_{j-1/2},t_k)}{\Delta y}+O(\Delta y) \tag{11}$$

$$\simeq G(|\nabla u^k_{i,j+1/2}|)(u^k_{i,j+1}-u^k_{i,j}) - G(|\nabla u^k_{i,j-1/2}|)(u^k_{i,j-1}-u^k_{i,j}) \tag{12}$$

we write finally

$$A(u(x_i,y_j,t_n)) \simeq A(u^n_{i,j}) \tag{13}$$

$$\simeq G_N.\nabla_N u^n_{i,j} + G_S.\nabla_S u^n_{i,j} + G_E.\nabla_E u^n_{i,j} + G_W.\nabla_W u^n_{i,j} \tag{14}$$

where:

$$\nabla_N u^n_{i,j} = u^n_{i-1,j} - u^n_{i,j}$$
$$\nabla_S u^n_{i,j} = u^n_{i+1,j} - u^n_{i,j}$$
$$\nabla_E u^n_{i,j} = u(^n_{i,j+1} - u^n_{i,j}$$
$$\nabla_W u^n_{i,j} = u^n_{i,j-1} - u^n_{i,j}$$

and

$$G_m = G\Bigg(|\nabla_m u(x_i, y_j, t)|\Bigg) \qquad\qquad m = N, S, E, W$$

C. Full Discretization

We will take an upper step in time for the fractional derivative in time
then we obtain an equivalence to our problem:

$$u^{k+1} = (1-c_1)u^k + \sum_{j=1}^{k-1} B_j u^{k-j} + c_k u^0 + \alpha_0 A(u^k) \quad (15)$$

and for $k = 0$

$$u^1 = \alpha_0 A(u^0) + u^0 \quad (16)$$

without forgetting the boundary condition and the initial condition

III. Experimental Results:

In this section, we give some experimental results which are obtained with application of the proposed model on images which have been corrupted by Gaussian noise, It is usually common in images acquired from cameras and telescopes, and it alters all pixels in the image.
Our proposed model is compared to the standard model of Perona-Malik, or/and classical heat equation.
We take $\Delta t = 0.1$ and $\Delta x = \Delta y = 1$
The number of iterations used is fixed in 17, and the performance of the models has been assessed by using PSNR(peak signal-to-noise ratio) and SNR(signal-to-noise ratio) which are defined by:

$$SNR = 10\log\left(\frac{\sum_{n,m} u^2}{\sum_{n,m}(\widehat{u}-u)^2}\right) \quad (17)$$

and

$$PSNR = 10\log\left(\frac{255^2}{\sum_{n,m} MSE}\right) \quad (18)$$

where u the original image and $\widehat{u}$ the restored image
and MSE is the Mean squared error given by

$$MSE = \frac{1}{MN}\sum_{n,m}(\widehat{u}-u)^2)$$

with M and N signifiant the width and height of the image
And for reasons of simplification we'll take $\lambda = 6$
Here we give a comparison between the proposed model and other models

A. First experience:

eyes original image in the left, noisy image with Gaussian noise with mean = 0 and variance $\sigma = 0.01$ in the middle, and noisy image with Gaussian noise with mean = 0 and $\sigma = 0,02$ in the right

result after 17 iterations for $\sigma = 0.01$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.63$

result after 17 iterations for $\sigma = 0.02$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.63$

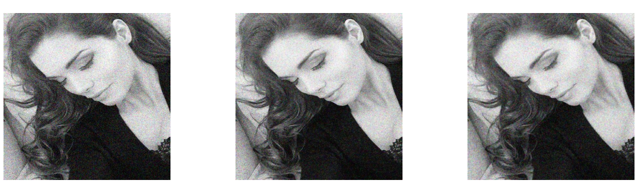
model	SNR	$PSNR$
heat equation	17.9148	23.1598
PM model	18.3240	23.5775
Proposed model with $\alpha = 0.63$	20.9576	26.2677
Proposed model with $\alpha = 0.81$	21.0049	26.8741

Table I: Performance of the proposed model for the first test with $\sigma = 0.01$

B. Second experience:

original image in the left, noisy image with Gaussian noise with mean = 0 and $\sigma = 0.01$ in the middle, and noisy image with Gaussian noise with mean = 0 and $\sigma = 0,02$ in the right

result after 17 iterations for $\sigma = 0.01$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.7$



result after 17 iterations for $\sigma = 0.02$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.7$

model	SNR	$PSNR$
heat equation	18.2883	23.6139
PM model	19.4911	24.8358
Proposed model with $\alpha = 0.7$	22.1066	27.4772

Table II: Performance of the proposed model for the second test with $\sigma = 0.01$

C. 3rd experience:

Lena ,original image in the left, noisy image with Gaussian noise with mean = 0 and $\sigma = 0.01$ in the middle, and noisy image with Gaussian noise with mean = 0 and $\sigma = 0,02$ in the right

result after 17 iterations for $\sigma = 0.01$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.7$

result after 17 iterations for $\sigma = 0.02$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.7$

model	SNR	$PSNR$
heat equation	14.7579	20.3170
PM model	15.7359	21.3160
Proposed model with $\alpha = 0.8$	19.9973	24.9802
Proposed model with $\alpha = 0.9$	19.0326	24.6910

Table III: Performance of the proposed model for the 3rd test with $\sigma = 0.02$

D. 4th experience:

Boat ,original image in the left, noisy image with Gaussian noise with mean = 0 and $\sigma = 0.01$ in the middle, and noisy image with Gaussian noise with mean = 0 and $\sigma = 0,02$ in the right

result after 17 iterations for $\sigma = 0.01$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.83$

result after 17 iterations for $\sigma = 0.02$, heat equation in the left, PM model in the middle and proposed method in the right with $\alpha = 0.83$

model	SNR	$PSNR$
heat equation	17.1683	23.2715
PM model	17.9907	24.1097
Proposed model with $\alpha = 0.83$	21.3823	27.4801
Proposed model with $\alpha = 0.9$	20.8720	27.0342

Table IV: Performance of the proposed model for the 4th test with $\sigma = 0.01$

model	SNR	$PSNR$
heat equation	14.4018	20.4189
PM model	15.2763	21.3265
Proposed model with $\alpha = 0.83$	18.6263	25.4392
Proposed model with $\alpha = 0.9$	18.3691	24.4915

Table V: Performance of the proposed model for the 4th test with $\sigma = 0.02$

IV. Conclusion

In this paper we have given a numerical scheme for the Perona-Malik model with fractional time derivative, we have applied the proposed model to images to which we have added Gaussian noise, we have compared the model with the heat equation and/or the classical Perona-Malik model, we have given the results and we have calculated the SNR and the PSNR it can be seen that the PSNR/SNR values of the proposed modelare higher than those of other models

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