EasyChair Preprint

№ 9029

# Riemann Hypothesis on Odd Perfect Numbers

Frank Vega



October 8, 2022

# Riemann hypothesis on odd perfect numbers

Frank Vega[1*]

[1*]Software Department, CopSonic, 1471 Route de Saint-Nauphary, Montauban, 82000, Tarn-et-Garonne, France.

Corresponding author(s). E-mail(s): vega.frank@gmail.com;

**Abstract**

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. The Riemann hypothesis is considered by many to be the most important unsolved problem in pure mathematics. Let $\sigma(n)$ denote the sum-of-divisors function $\sigma(n) = \sum_{d \mid n} d$. An integer $n$ is perfect if $\sigma(n) = 2 \cdot n$. It is unknown whether any odd perfect numbers exist. Leonhard Euler stated: "Whether $\ldots$ there are any odd perfect numbers is a most difficult question". We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. We also use Robin's criterion and Ramanujan's old notes which were published in 1997 annotated by Jean-Louis Nicolas and Guy Robin. There are several statements equivalent to the famous Riemann hypothesis. In this note, conditional on Riemann hypothesis, we prove that there is no odd perfect number.

**Keywords:** Riemann hypothesis, Odd perfect numbers, Superabundant numbers, Sum-of-divisors function, Prime numbers

**MSC Classification:** 11M26 , 11A41 , 11A25

## 1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of $n$

$$\sum_{d \mid n} d,$$

## 2 *The Riemann hypothesis*

where $d \mid n$ means the integer $d$ divides $n$. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

**Proposition 1** *For every prime power $p^a$, we have $f(p^a) = \frac{p^{a+1}-1}{p^a \cdot (p-1)}$ [1, Lemma 2.1 pp. 3]. If $p$ is a prime number, and $a$, $b$ two positive integers, then [1, pp. 5]:*

$$f(p^{a+b}) - f(p^a) \cdot f(p^b) = -\frac{(p^a - 1) \cdot (p^b - 1)}{p^{a+b-1} \cdot (p-1)^2}.$$

We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The Ramanujan's Theorem states that if the Riemann hypothesis is true, then the previous inequality holds for large enough $n$. Next, we have the Robin's Theorem:

**Proposition 2** $\mathsf{Robin}(n)$ *holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [2, Theorem 1 pp. 188].*

In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [3]. Let $p_1 = 2, p_2 = 3, \ldots, p_k$ denote the first $k$ consecutive primes, then an integer of the form $\prod_{i=1}^{k} p_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer. A natural number $n$ is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following properties for the superabundant numbers:

**Proposition 3** *If $n$ is superabundant, then $n$ is a Hardy-Ramanujan integer [4, Theorem 1 pp. 450].*

**Proposition 4** *[4, Theorem 7 pp. 454]. Let $n$ be a superabundant number such that $p$ is the largest prime factor of $n$, then*

$$p \sim \log n, \quad (n \to \infty).$$

**Proposition 5** *[4, Theorem 9 pp. 454]. For some constant $c > 0$, the number of superabundant numbers less than $x$ exceeds*

$$\frac{c \cdot \log x \cdot \log \log x}{(\log \log \log x)^2}.$$

A number $n$ is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \quad \textit{for} \quad (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

**Proposition 6** *Every colossally abundant number is superabundant [4, pp. 455].*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true.

**Proposition 7** *Ramanujan [3] proved that if $n$ is a generalized superior highly composite number, i.e., a colossally abundant number, then under the Riemann hypothesis we have*

$$\varliminf_{n\to\infty} \big(f(n) - e^{\gamma} \cdot \log\log n\big) \cdot \sqrt{n} \geq -e^{\gamma} \cdot (2 \cdot \sqrt{2} + \gamma - \log(4 \cdot \pi)) \approx -1.558.$$

In number theory, the *p-adic* order of an integer $n$ is the exponent of the highest power of the prime number $p$ that divides $n$. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which $p$ appears in the prime factorization of $n$.

**Proposition 8** *[5, Theorem 4.4 pp. 12]. Let $n$ be a superabundant number such that $p$ is the largest prime factor of $n$ and $2 \leq p \leq p$, then*

$$\left\lfloor \frac{\log p}{\log p} \right\rfloor \leq \nu_p(n).$$

In mathematics, a perfect number is a positive integer $n$ such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{a-1} \cdot (2^a - 1)$ whenever $2^a - 1$ is prime. It is unknown whether any odd perfect numbers exist, but under the assumption that the Riemann hypothesis is true, we prove that there is no odd perfect number.

## 2 Main Insight

**Lemma 9** *For every odd prime $p$ and two natural numbers $a$, $b$, we have*

$$f(p^a) \cdot f(p^b) \leq 1.5 \cdot f(p^{a+b}).$$

*Proof* We know that

$$f(p^a) \cdot f(p^b) - f(p^{a+b}) = \frac{(p^a - 1) \cdot (p^b - 1)}{p^{a+b-1} \cdot (p-1)^2}$$

by Proposition 1. Hence, it is enough to show that

$$\frac{(p^a-1)\cdot(p^b-1)}{p^{a+b-1}\cdot(p-1)^2} \leq \frac{f(p^{a+b})}{2}$$

which is the same as

$$\frac{(p^a-1)\cdot(p^b-1)}{p^{a+b-1}\cdot(p-1)^2} \leq \frac{p^{a+b+1}-1}{p^{a+b}\cdot(p-1)\cdot 2}$$

by Proposition 1. That is equivalent to

$$\frac{p\cdot(p^a-1)\cdot(p^b-1)}{p-1} \leq \frac{p^{a+b+1}-1}{2}.$$

For every odd prime $p$,

$$p-1 \geq 2$$

and

$$\begin{aligned} p\cdot(p^a-1)\cdot(p^b-1) &= (p^{a+1}-p)\cdot(p^b-1) \\ &= p^{a+b+1}-p^{a+1}-p^{b+1}+p \\ &\leq p^{a+b+1}-1 \end{aligned}$$

and thus, the inequality

$$\frac{p\cdot(p^a-1)\cdot(p^b-1)}{p-1} \leq \frac{p^{a+b+1}-1}{2}$$

holds. □

# 3 Main Theorem

This is the main theorem.

**Theorem 10** *Under the assumption that the Riemann hypothesis is true, we prove that there is no odd perfect number.*

*Proof* Suppose that $N$ is the smallest odd perfect number, then we will show this implies that the Riemann hypothesis should be false. We deduce there are infinitely many colossally abundant numbers by Propositions 5 and 6. Let $n$ be a large enough colossally abundant number. Under the assumption that the Riemann hypothesis is true and by definition of limit inferior, a value of $b \geq -e^{\gamma}\cdot(2\cdot\sqrt{2}+\gamma-\log(4\cdot\pi)) \approx -1.558$ is the largest real number such that, for any positive real number $\varepsilon$, there exists a natural number $m$ such that

$$x_n > b-\varepsilon$$

for all $n > m$, where

$$x_n = \left(f(n)-e^{\gamma}\cdot\log\log n\right)\cdot\sqrt{n}$$

is a sequence of real numbers by Proposition 7. Since $n$ is large enough, then

$$x_n > -1.559$$

could be possible, which is the same as

$$f(n) > e^{\gamma} \cdot \log\log n - \frac{1.559}{\sqrt{n}}.$$

On the other hand, we can see that the value of $\nu_2(n)$ goes to infinity as long as $n$ goes to infinity since $n$ is colossally abundant according to Propositions 3, 4, 6 and 8. So, the number $2^{\nu_2(n)}$ could be too much greater than $N$ since $N$ is a fixed natural number. We know that

$$\begin{aligned} f(n) &= f(2^{\nu_2(n)}) \cdot f(\frac{n}{2^{\nu_2(n)}}) \\ &= \left(2 - \frac{1}{2^{\nu_2(n)}}\right) \cdot f(\frac{n}{2^{\nu_2(n)}}) \\ &= 2 \cdot f(\frac{n}{2^{\nu_2(n)}}) - \frac{f(\frac{n}{2^{\nu_2(n)}})}{2^{\nu_2(n)}} \\ &= f(N) \cdot f(\frac{n}{2^{\nu_2(n)}}) - \frac{f(\frac{n}{2^{\nu_2(n)}})}{2^{\nu_2(n)}} \\ &\leq 1.5 \cdot f(\frac{N \cdot n}{2^{\nu_2(n)}}) - \frac{f(\frac{n}{2^{\nu_2(n)}})}{2^{\nu_2(n)}} \\ &= 1.5 \cdot f(n') - \frac{f(\frac{n'}{N})}{2^{\nu_2(n)}} \end{aligned}$$

by Proposition 1 and Lemma 9, since $f(\ldots)$ is multiplicative and $n' = \frac{N \cdot n}{2^{\nu_2(n)}}$. Therefore, we have

$$1.5 \cdot f(n') - \frac{f(\frac{n'}{N})}{2^{\nu_2(n)}} > e^{\gamma} \cdot \log\log n - \frac{1.559}{\sqrt{n}}$$

which is the same as

$$1.5 \cdot f(n') > e^{\gamma} \cdot \log\log n - \frac{1.559}{\sqrt{n}} + \frac{f(\frac{n'}{N})}{2^{\nu_2(n)}}.$$

Let subtract by $1.5 \cdot e^{\gamma} \cdot \log\log n'$ both sides to obtain that

$$\left(f(n') - e^{\gamma} \cdot \log\log n'\right) > \left(\frac{2}{3} \cdot \left(e^{\gamma} \cdot \log\log n - \frac{1.559}{\sqrt{n}} + \frac{f(\frac{n'}{N})}{2^{\nu_2(n)}}\right) - e^{\gamma} \cdot \log\log n'\right).$$

We know that

$$\left(\frac{2}{3} \cdot \left(e^{\gamma} \cdot \log\log n - \frac{1.559}{\sqrt{n}} + \frac{f(\frac{n'}{N})}{2^{\nu_2(n)}}\right) - e^{\gamma} \cdot \log\log n'\right) > 0$$

because of $n$ could be too much greater than $n'$. Since we took an arbitrary large enough colossally abundant number, then there are infinitely many natural numbers $n'$ such that $\mathsf{Robin}(n')$ does not hold. Hence, the Riemann hypothesis must be necessarily false under the assumption of the existence of such smallest odd perfect number $N$ by Proposition 2. In this way, the number $N$ does not exist under the assumption that the Riemann hypothesis is true. □

**Acknowledgments.** The author specially wishes to thank Patrick Solé for being his informal advisor on this topic. The author also wishes to thank his mother, maternal brother, maternal aunt, and friends Liuva, Yary, Sonia, and Arelis for their support.

## References

[1] R. Vojak, On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds. arXiv preprint arXiv:2005.09307v1 (2020)

[2] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. pures appl **63**(2), 187–213 (1984)

[3] J.L. Nicolas, G. Robin, Highly Composite Numbers by Srinivasa Ramanujan. The Ramanujan Journal **1**(2), 119–153 (1997). https://doi.org/10.1023/A:1009764017495

[4] L. Alaoglu, P. Erdős, On Highly Composite and Similar Numbers. Transactions of the American Mathematical Society **56**(3), 448–469 (1944). https://doi.org/10.2307/1990319

[5] S. Nazardonyavi, S. Yakubovich, Superabundant numbers, their subsequences and the Riemann hypothesis. arXiv preprint arXiv:1211.2147v3 (2013)