Static Bending Analysis of FGM Stiffened Plate Resting on Discontinuous Elastic Foundation

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Abstract. This paper uses the finite element method to study the static bending response of stiffened FGM plates resting on a discontinuous elastic foundation. The structure is computed using the first-order shear deformation theory coupled with the four-node and two-node quadrilateral elements. The accuracy of the method is evaluated by validating the obtained numerical results against reference solutions available in the literature. In addition, the effect of certain geometrical parameters, materials, and elastic foundations on the static bending response of the plate is also researched. This is a reference for the design and practical use of stiffened structures.

Keywords: FGM stiffened plate, discontinuous elastic foundation, static bending.

1. Introduction

Functionally Graded Composite Materials (FGM) are made from a mixture of ceramic and metal in certain proportions. Due to both ceramic and metal properties, especially heat resistance, and anti-radiation, the application of these materials can be used widely in aerospace, marine, civil construction, nuclear reactor industry, and so on [1-10]. In addition, the mechanical behavior of plates will change in the presence of a foundation. Plates supported by the ground, water, and some kinds of liquid can be modeled as a plate resting on an elastic foundation. Developing a realistic mathematical model for this complex foundation-structure interaction problem is essential to provide safe and economical designs [13].

Because of FGM plate advantages, many researchers have focused on the mechanical analysis of these structures in the past decades. Duc and Cong [11] applied the Galerkin method and stress function for nonlinear post-buckling of discontinuous eccentrically stiffened thin FGM plates resting on the Pasternak elastic foundation. Thai and his co-worker [12] used the third-order shear deformation theory to establish the closed-form solution of thick FGM plates supported by the Pasternak foundation. So the problem of the plate resting on an elastic foundation is very useful in the design of FGM for engineering applications. This study aims to present a finite element algorithm for static analysis of FGM stiffened plates resting on discontinuous elastic foundations.

2. The theoretical model and governing equations.

Considering a rectangular stiffened plate satisfying the Mindlin hypothesis (Figure 1). The material plate is made from a mixture of metal and ceramic components. The material properties vary continuously through the plate thickness by an exponential function. Stiffeners are made of the same material as the plate surface material on which it is placed.

In this study, a rectangular FGM stiffened plate resting on a discontinuous elastic foundation as shown in Figure 1 is considered, and the dimension of the plate is $a\times b\times h$. The volume fraction proportion is calculated by the following law:

$$V_m + V_c = 1 \text{ and } V_c(z) = \left(\frac{2z+h}{2h}\right)^n \quad (1)$$

where z is the thickness coordinate variable, n is the volume fraction exponent $(n \geq 0)$; V_m and V_c are the volume fractions of the metal and ceramic, respectively.

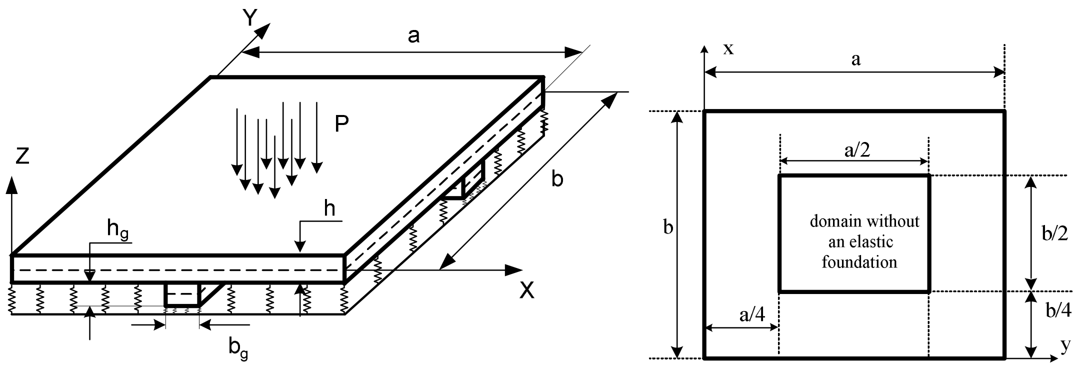

Figure. 1. Geometry of rectangular FGM plate resting on a discontinuous elastic foundation.

3. Finite element formulation for mechanical behavior of the plate.

According to [4], the displacement field can be expressed as follows:

$$u = u_0 + z\varphi_x;\ v = v_0 + z\varphi_y;\ w = w_0 \quad (2)$$

where u_0, v_0, w_0 are the displacements at the mid-plane of a plate in the x, y, z directions and φ_x, φ_y are the transverse normal rotations of the y and x axes, respectively.

In this study, we adopted a quadrilateral four-node plate element; the strain field, stress field, and internal force can be given as in [3].

4. Finite element formulation for mechanical behavior of the stiffeners.

4.1. The x -direction stiffener element

In this work, the stiffener was assumed to be parallel to the x-axis and y-axis (See Figure 2). The displacement component for the x-stiffener element can be expressed as:

$$u_{xg} = u_{0xg} + z\varphi_{xg};\ \ v_{xg} = 0;\ \ w_{xg} = w_{0xg} \quad (3)$$

where $u_{0xg}, w_{0xg}, \varphi_{xg}$ are the displacements in the middle surface and the rotation around the y-axis, respectively.

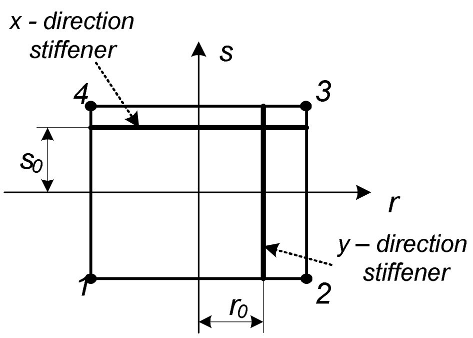

Figure 2. Four-node FGM plate element with stiffener parallel to the plate edges

The displacement of an arbitrary stiffener element will be expressed by the shape function and the node displacement vector:

$$\left\{u_{xg}\right\}_e=\left\{u_{xg}\ v_{xg}\ w_{xg}\ \varphi_{xg}\ \varphi_{yg}\right\}_e^T=\sum_{i=1}^{4}N_i\left[I\right]_{5x5}\left\{q_{ixg}\right\}_e=\left[N\right]\left\{q_{xg}\right\}_e \tag{4}$$

where N_i is the shape function, which can be expressed in terms of the plate by substituting $s = s_{0;}$ $\left\{q_{ixg}\right\}_e=\left\{u_{0ixg}\ v_{0ixg}\ w_{0ixg}\ \varphi_{ixg}\ \varphi_{iyg}\right\}_e^T$ - nodal displacement vector i.

The strain field:

$$\left\{\varepsilon_{xg}\right\}_e=\left\{\varepsilon_{mxg}\right\}_e+z\left\{\kappa_{xg}\right\}_e=\left[B_{1xg}\right]\left\{q_{xg}\right\}_e+z\left[B_{2xg}\right]\left\{q_{xg}\right\}_e;\left\{\gamma_{xg}\right\}_e=\left[B_{3xg}\right]\left\{q_{xg}\right\}_e \tag{5}$$

where $\left[B_{1xg}\right]$, $\left[B_{2xg}\right]$, $\left[B_{3xg}\right]$ are the differential shape functions, respectively.

Relation between the stress and strain is expressed as follows:

$$\sigma_{xg}=E\varepsilon_{xg};\sigma_{yg}=E\varepsilon_{yg};\tau_{xzg}=\frac{E}{2\left(1+\nu\right)};\tau_{yzg}=\frac{E}{2\left(1+\nu\right)} \tag{6}$$

Stiffeners' internal force field is calculated as:

$$\left(\left\{N_{xg}\right\}_e,\left\{M_{xg}\right\}_e\right)=\int_{-h_g/2}^{h_g/2}\begin{Bmatrix}\sigma_{xg}\\0\\0\end{Bmatrix}_e(1,z)dz=\int_{-h_g/2}^{h_g/2}\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_{xg}\\0\\0\end{Bmatrix}_e(1,z)dz \tag{7}$$

$$\left\{Q_{xg}\right\}_e=\begin{Bmatrix}Q_{xg}\\Q_{yg}\end{Bmatrix}_e=\int_{-h_g/2}^{h_g/2}\begin{Bmatrix}\tau_{xg}\\0\end{Bmatrix}_e zdz=\int_{-h_g/2}^{h_g/2}\frac{5E}{12(1+\nu)}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{Bmatrix}\gamma_{xg}\\0\end{Bmatrix}_e dz \tag{8}$$

with h_g is the height of the stiffeners.

According to the displacement compatibility condition, one has:

$$\begin{Bmatrix}u_{0xg}\\v_{0xg}\\w_{0xg}\\\varphi_{xg}\\\varphi_{yg}\end{Bmatrix}=\begin{bmatrix}1&0&0&e_{tg}&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}\begin{Bmatrix}u_0\\v_0\\w_0\\\varphi_x\\\varphi_y\end{Bmatrix} \tag{9}$$

Equation (9) can be rewritten in the matrix form as: $\left\{u_{xg}\right\}=\left[T_{xg}\right]\left\{u\right\}$ where $\left\{q_{xg}\right\}_e=\left[T_x^{tg}\right]\left\{q\right\}_e$.

4.2. The y-direction stiffener element

Displacement components of y-direction stiffeners are defined as follows (Figure 2):

$$u_{yg}=0;\quad v_{xg}=u_{0yg}+z\varphi_{yg};\quad w_{yg}=w_{0yg} \tag{10}$$

where $v_{0yg},w_{0yg},\varphi_{yg}$ are the displacements on the middle surface and the rotation around the x-axis, respectively.

The same calculation as x-direction stiffeners, the displacement compatibility condition between stiffeners and plate is expressed as follows:

$$\left\{q_{yg}\right\}_e=\left[T_y^{tg}\right]\left\{q\right\}_e \tag{11}$$

with

$$\left[T_y^{tg}\right]=\begin{bmatrix}\left[T_{yg}\right] & [0] & [0] & [0]\\ [0] & \left[T_{yg}\right] & [0] & [0]\\ [0] & [0] & \left[T_{yg}\right] & [0]\\ [0] & [0] & [0] & \left[T_{yg}\right]\end{bmatrix} \tag{12}$$

4.3. The foundation element

In this study, the plate is placed on a two-parameter (k_w and k_s) elastic foundation, then the elastic foundation's strain energy in a plate element is computed as follows:

$$U_e^f=\frac{1}{2}\int_{V_e}\left(k_w w^2+k_s\left(\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)+\left(\frac{\partial w}{\partial y}\right)^T\left(\frac{\partial w}{\partial y}\right)\right)\right)dV_e \tag{13}$$

By expanding similar to the above expressions, the additional stiffness of the elastic foundation is obtained as follows:

$$\left[K^f\right]_e=\int_S\left(k_w\left[\mathrm{N}_w\right]^T\left[\mathrm{N}_w\right]+k_s\left(\left[\mathrm{N}_{w,x}\right]^T\left[\mathrm{N}_{w,x}\right]+\left[\mathrm{N}_{w,\mathrm{y}}\right]^T\left[\mathrm{N}_{w,\mathrm{y}}\right]\right)\right)dS \tag{14}$$

where

$$\left[\mathrm{N}_w\right]=\sum_1^4\left[0,0,N_i,0,0\right];\left[\mathrm{N}_{w,x}\right]=\sum_1^4\left[0,0,\frac{\partial N_i}{\partial x},0,0\right];\left[\mathrm{N}_{w,\mathrm{y}}\right]=\sum_1^4\left[0,0,\frac{\partial N_i}{\partial y},0,0\right] \tag{15}$$

5. Matrix and static equilibrium equation of the structure

The elastic potential energy of the plate element including the elastic foundation and the stiffeners is expressed as follows:

$$U_e=\frac{1}{2}\{q\}_e^T\left(\left[K_t\right]_e+\left[K_{xg}\right]_e+\left[K_{yg}\right]_e\right)\{q\}_e-\{q\}_e^T\{F\}_e+\text{const} \tag{16}$$

where $\left[K_t\right]_e$, $\left[K_{xg}\right]_e$, $\left[K_{yg}\right]_e$ are the plate element stiffness matrix, the x-direction stiffener element stiffness matrix, the y-direction stiffener element stiffness matrix, and $\{F\}_e$ is the element nodal force vector due to uniformly distributed external force acting perpendicular to the mid-plane. These matrixes can be given as:

$$\left[K_t\right]_e=\int_{S_e}\left(\left[B_1\right]^T[A]\left[B_1\right]+\left[B_1\right]^T[B]\left[B_2\right]+\left[B_2\right]^T[B]\left[B_1\right]+\left[B_2\right]^T[D]\left[B_2\right]+\left[B_3\right]^T[A']\left[B_3\right]\right)dS_e \tag{17}$$

$$\left[K_{xg}\right]_e=\left[T_x^{tg}\right]^T\left(b_{xg}\int_{l_{xe}}\left(\left[B_{1xg}\right]^T\left[A_g\right]\left[B_{1xg}\right]+\left[B_{2xg}\right]^T\left[B_g\right]\left[B_{2xg}\right]+\left[B_{3g}\right]^T\left[A_g{}'\right]\left[B_{3xg}\right]\right)dx\right)\left[T_x^{tg}\right] \tag{18}$$

$$\left[K_{yg}\right]_e = \left[T_y^{tg}\right]^T \left(b_{yg} \int_{l_{ye}} \left(\left[B_{1yg}\right]^T \left[A_g\right]\left[B_{1yg}\right] + \left[B_{2yg}\right]^T \left[B_g\right]\left[B_{2yg}\right] + \left[B_{3yg}\right]^T \left[A_g{}'\right]\left[B_{3yg}\right] \right) dy \right) \left[T_y^{tg}\right] \quad (19)$$

$$\{F\}_e = \int_{S_e} [N]^T \{P\} dS_e \quad (20)$$

where P is uniformly distributed load acting perpendicular to the mid-plane, b_{xg}, b_{yg} - the width of stiffeners parallel to the x, y axes. Integrals in the expression (17) ÷ (20) are calculated by the Gaussian quadrature method [4]. According to the principle of minimum total potential energy, conduct matrix aggregation, remove boundary conditions [3], and the governing equation of the plate is obtained as:

$$[K]\{q\} = \{F\} \quad (21)$$

Equation (21) is solved in the Matlab environment.

6. Numerical results and discussion

6.1. Accuracy Study

To confirm the accuracy and reliability of this approach, consider an un-stiffened rectangular FGM plate with a length of $a = b = 1$ m, and a thickness of $h = a/10$. The plate was simply supported at all edges. The material properties are [8, 9]: Aluminum (metal): E_m = 70.10^9 (N/m^2), $\nu_m = 0,3$; Alumina (ceramic): E_c = 380.10^9 (N/m^2), $\nu_c = 0,3$. The load P is uniformly distributed over the plate. The results of the maximum deflection at the center of the plate compared with the analytical results of Zenkour [5] are shown in Table 1.

Table 1. The maximum static displacement of the plate

	[5]	Present
$\overline{w} = 10w\left(\frac{a}{2},\frac{b}{2}\right)E_c h^3/(Pa^4)$	1,194	1,198
Error	0,02%	

6.2. Effect of the stiffeners on the static response of the FGM plate

Table 2. Maximum deflection w_{max} and stress $|\sigma_x|$ max

	Un - stiffened FGM plate	FGM plate with 1 stiffener	FGM plate with 2 stiffeners
Simply supported plate			
w_{max} (m)	1,30.10^{-3}	0,93.10^{-3}	0,73.10^{-3}
$\sigma_{x\max}$ (N/m^2)	1,742.10^8	1,306.10^8	1,010.10^8
Clamped plate			
w (m)	1,21.10^{-5}	0,87.10^{-5}	0,68.10^{-5}
$\sigma_{x\max}$ (N/m^2)	2,916.10^6	1,709.10^6	1,133.10^6

Consider Al_2O_3/SUS304 plate with the parameter $a = b$ = 1m, the thickness $h = a/10$, and the volume fraction exponent n = 0,5. The properties of the functionally graded material components [6]:

Ceramic: $E_c = 320,24.10^9$ (Pa), $\nu_c = 0,260$, $\rho = 3800$ (kg/m^3);

Metal: $E_m = 207,79.10^9$ (Pa), $\nu_m = 0,318$, $\rho = 8166$ (kg/m^3);

Consider the problem in several cases: An un-stiffened plate, a plate with one central stiffener, and a plate with two central stiffeners at the metal-rich surface. The stiffener width $b_g = 2$ cm, the stiffener height $h_g = 2h$. The plate was simply supported or claimed at all edges. A uniformly and perpendicularly distributed load on the ceramic-rich plate surface has a value of 2.10^5 N/m^2.

Using the established program, the displacement and stress values $w_{\max} = w(a/2, b/2)$; $\sigma_{x\max} = \sigma_x(a/2, b/2, h/2)$ at the top surface of the mid-plate are presented in Table 2.

The maximum displacement and stress of the FGM stiffened plate have been greatly reduced compared to the FGM non-stiffened plate, the more stiffeners the plate has, the more displacement and stress will be reduced. Thus, stiffeners have a significant effect on increasing the stiffness and strength of the FGM plate.

6.3. Effect of material volume ratio of ceramic and metal

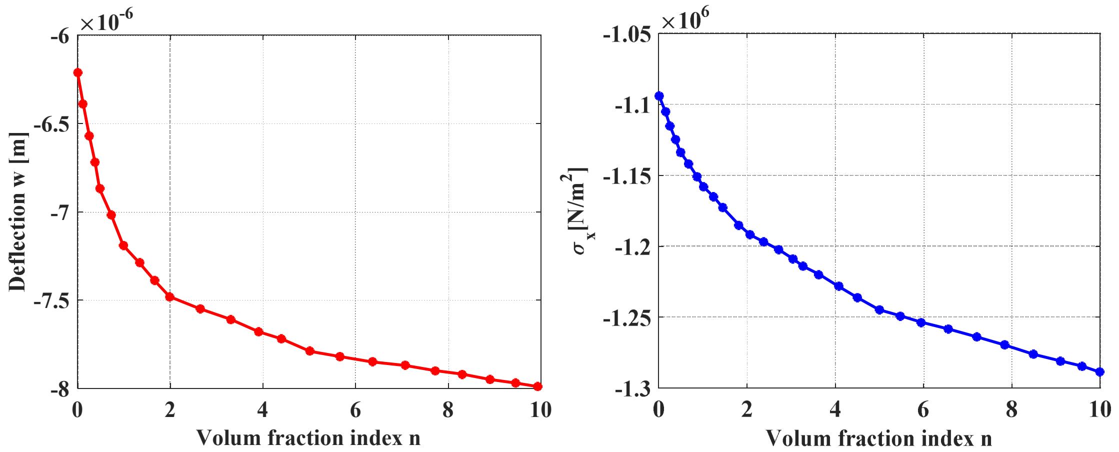

Figure 3. Effect of volume fraction exponent on deflection and stress at the midpoint of the FGM plate.

Considering a clamped FGM plate of the same size as above, two perpendicular stiffeners at the center of the plate. Changing the volume fraction exponent n, the graphical investigation representing the variation of deflection and stress σ_x at the midpoint ($x=a/2$; $y=b/2$; $z=h/2$) of the plate is shown in Figure 3.

As the exponent n increases from 0 to 10, the displacement and stress at the midpoint σ_x of the plate increase, which shows that the “stiffness” of the plate in bending decreases, and the variation of both deflection and stress is high. when n varies in the range of 0 - 2.

7. Conclusions

In this paper, we have built the finite element algorithm and examined some specific examples to determine the deflection and stress at the midpoint of the FGM stiffened plate resting on a discontinuous elastic foundation. The results show that, with the specific problems considered above, when reducing the number of stiffeners, and increasing the volume fraction exponent n, the deflection and stress in the middle of the plate increase. Therefore, depending on actual requirements, the above parameters can be reasonably selected when designing FGM stiffened plates to obtain the most reasonable structural form to ensure the required strength and durability.

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