About the M|G|∞ Queue Approximation by a Markov Renewall Process

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Abstract: Some $M|G|\infty$ queue systems interesting quantities values approximations, obtained through the consideration of an adequate Markov renewal process, are presented, and discussed.

Keywords: $M|G|\infty$*;* Markov renewal process; queue.

1. Introduction

In the $M|G|\infty$ queue system the customers arrive according to a Poisson process at rate λ, receive a service which time is a positive random variable with distribution function $G(.)$ and mean α and, when they arrive, find immediately an available server. Each customer service is independent from the other customers' services and from the arrivals process. The traffic intensity is $\rho = \lambda\alpha$.

A suggestion to obtain approximate results for these systems, when the exact ones are still not known, is to use a Markov renewal process, see (1,2).

Along this work, some of the approximations so obtained are reviewed.

2. Sojourn Time Mean Value in State *k*

For the process referred above, the sojourn time mean value in state[1] $k, k = 0,1, \ldots$ is given by:

$$m_k = \int_0^\infty e^{-\lambda t} \left[\frac{\int_t^\infty [1 - G(x)]dx}{\alpha}\right]^k dt, k = 0,1, \ldots \qquad (2.1).$$

Proposition 2.1

$$m_0 = \frac{1}{\lambda} \qquad (2.2).$$

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[1] The state of the $M|G|\infty$ queue in instant *t* is the number of customers being served in the system at instant *t*.

Obs.: The sojourn time mean value in state 0, does not depend on $G(.)$. It depends only on the arrivals process rate.

Proposition 2.2

$m_k \leq \frac{1}{\lambda}$, $k = 0,1,...$ (2.3).

Dem: It is enough to note that $\alpha^{-\mathbf{1}} \int_t^{\infty} [1 - G(x)]dx \leq 1$. □

Obs: So, the sojourn time mean value in any state does not exceed the sojourn time in state 0. Define:

$$E_0 = \frac{1}{\lambda} \text{ (2.4).}$$

Proposition 2.3

$$m_k \leq \alpha \sqrt{\frac{\gamma_s^2 + 1}{2\rho(2k + 1)}},, k = 1,2,... \quad (2.5)$$

Being γ_s the service coefficient of variation.

Dem: Using the Schwartz's inequality, $m_k^2 \leq \int_0^{\infty} e^{-2\lambda t} dt \int_0^{\infty} \left[\frac{\int_t^{\infty}[1-G(x)]dx}{\alpha}\right]^{2k} dt = \frac{1}{2\lambda\alpha^{2k}} \int_0^{\infty} \left[\int_t^{\infty} [1 - G(x)]dx\right]^{2k} dt = \frac{1}{2\lambda\alpha^{2k}} \frac{2k\alpha^2}{2} (\gamma_s^2 + 1) \frac{\alpha^{2k-1} b_{2k-1}}{2k(2k+1)} \leq \alpha \frac{\gamma_s^2+1}{2\lambda(2k+1)}$
since, see (3),

$$\int_0^{\infty} \left[\int_t^{\infty} [1 - G(x)]dx\right]^n dt = \frac{n\alpha^2}{2} (\gamma_s^2 + 1) \frac{\alpha^{n-1} b_{n-1}}{n(n + 1)} \text{with} b_n \leq 2,$$
$$n = 0,1,... \quad (2.6). \square$$

Obs: Define

$$E_1 = \alpha \sqrt{\frac{\gamma_s^2 + 1}{2\rho(2k + 1)}} \text{(2.7).}$$

Proposition 2.4

If $k \geq \frac{1}{4} \rho(\gamma_s^2 + 1) - \frac{1}{2}, E_1 \leq E_0$.

Dem:

$$\alpha \sqrt{\frac{\gamma_s^2 + 1}{2\rho(2k + 1)}} \leq \frac{1}{\lambda} \Leftrightarrow \frac{\gamma_s^2 + 1}{2\rho(2k + 1)} \leq \frac{1}{\rho^2} \Leftrightarrow \rho(\gamma_s^2 + 1) \leq 4k + 2 \Leftrightarrow k \geq \frac{1}{4} \rho(\gamma_s^2 + 1) - \frac{1}{2}. \square$$

Proposition 2.5

$$m_k \leq \alpha \frac{\gamma_s^2 + 1}{k + 1}, k = 1,2,... \quad (2.8).$$

Dem:

$$m_k \le \int_0^\infty \left[\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right]^k dt \le \frac{1}{\alpha^k}\frac{k\alpha^2}{2}(\gamma_s^2+1)\frac{2\alpha^{k-1}}{k(k+1)} = \alpha\frac{\gamma_s^2+1}{k+1}$$

After (2.6). □

Obs: Define

$$E_2 = \alpha\frac{\gamma_s^2+1}{k+1}(2.9).$$

Proposition 2.6

If $k \ge \rho(\gamma_s^2+1)-1, E_2 \le E_0$.

Dem:

$$\alpha\frac{\gamma_s^2+1}{k+1} \le \frac{1}{\lambda} \Leftrightarrow \rho(\gamma_s^2+1) \le k+1 \Leftrightarrow k \ge \rho(\gamma_s^2+1)-1. \square$$

Proposition 2.7

If $k \le 2\rho(\gamma_s^2+1)-1, E_1 \le E_2$.

Dem:

$$\frac{E_1}{E_2} = \frac{\alpha\sqrt{\frac{\gamma_s^2+1}{2\rho(2k+1)}}}{\alpha\frac{\gamma_s^2+1}{k+1}} = \frac{\sqrt{\gamma_s^2+1}(k+1)}{(\gamma_s^2+1)\sqrt{2\rho(2k+1)}} = \frac{k+1}{\sqrt{\gamma_s^2+1}\sqrt{2\rho}\sqrt{2k+1}}$$

$$\le \sqrt{\frac{k+1}{2\rho(\gamma_s^2+1)}}\,;\frac{k+1}{2\rho(\gamma_s^2+1)} \le 1 \Leftrightarrow k+1 \le 2\rho(\gamma_s^2+1) \Leftrightarrow k$$
$$\le 2\rho(\gamma_s^2+1)-1. \square$$

Proposition 2.8

If $k \ge 4\rho(\gamma_s^2+1)-1, E_2 \le E_1$.

Dem:

$$\frac{E_1}{E_2} = \frac{k+1}{\sqrt{2\rho(\gamma_s^2+1)}\sqrt{2k+1}} \ge \frac{k+1}{\sqrt{4\rho(\gamma_s^2+1)}\sqrt{k+1}} = \sqrt{\frac{k+1}{4\rho(\gamma_s^2+1)}}\,;\frac{k+1}{4\rho(\gamma_s^2+1)}$$
$$\ge 1 \Leftrightarrow k \ge 4\rho(\gamma_s^2+1)-1. \square$$

The Propositions 2.4, 2.6, 2.7 and 2.8 lead to the following upper bounds choice for $m_k, k = 1,2,\dots$:

A) $\rho(\gamma_s^2+1) > \frac{2}{3}$

$$k < \frac{1}{4}\,\rho(\gamma_s^2+1) - \frac{1}{2} m_k \le \frac{1}{\lambda}$$

$$\frac{1}{4}\,\rho(\gamma_s^2+1) - \frac{1}{2} \le k \le 2\rho(\gamma_s^2+1) - 1 \qquad m_k \le \alpha\sqrt{\frac{\gamma_s^2+1}{2\rho(2k+1)}}$$

$$2\rho(\gamma_s^2+1) - 1 < k < 4\rho(\gamma_s^2+1) - 1 \qquad m_k \le min\left\{\alpha\sqrt{\frac{\gamma_s^2+1}{2\rho(2k+1)}}, \alpha\frac{\gamma_s^2+1}{k+1}\right\}$$

$$k \ge 4\rho(\gamma_s^2+1) - 1 \qquad m_k \le \alpha\frac{\gamma_s^2+1}{k+1}$$

B) $\frac{1}{2} < \rho(\gamma_s^2+1) \le \frac{2}{3}$

$$k = 1 \qquad m_1 \le min\left\{\alpha\sqrt{\frac{\gamma_s^2+1}{6\rho}}, \alpha\frac{\gamma_s^2+1}{2}\right\}$$

$$k = 2,3,\ldots \qquad m_k \le \alpha\frac{\gamma_s^2+1}{k+1}$$

C) $\rho(\gamma_s^2+1) \le \frac{1}{2}$

$$m_k \le \alpha\frac{\gamma_s^2+1}{k+1}, k = 1,2,\ldots$$

Proposition 2.9

If the service time distribution is *NBUE*

$$m_k \le \frac{\alpha}{k+\rho}, k = 1,2,\ldots \qquad (2.10)$$

Dem: It is enough to note that if the service time is *NBUE* with mean α, $\int_b^\infty [1 - G(x)]dx \le \int_b^\infty e^{-\frac{x}{\alpha}}dx$, for any $b \ge 0$. □

Obs: If the service time is *NWUE* with mean α, $\int_b^\infty [1 - G(x)]dx \ge \int_b^\infty e^{-\frac{x}{\alpha}}dx$, for any $b \ge 0$ and

$$m_k \ge \frac{\alpha}{k+\rho}, k = 1,2,\ldots \qquad (2.11).$$

Proposition 2.10

If the service time distribution is *IMRL*

$$m_k \geq e^{k\left(1-\frac{2\,\alpha}{3\mu_2^2}\mu_3\right)}\frac{\mu_2}{\mu_2\lambda+2k\alpha}, k = 1,2,\ldots \qquad (2.12)$$

beingμ_2 and μ_3 the 2nd and the 3rd$G(.)$ moments around the origin

Dem: If the service time[2] is *IMRL*

$1-G^*(x)=1-\frac{1}{\alpha}\int_0^x[1-G(y)]dy = \frac{\int_x^\infty[1-G(y)]dy}{\alpha} \geq e^{-\frac{2\alpha}{\mu_2}x-\frac{2\,\alpha}{3\mu_2^2}\mu_3+1}$.

So, $m_k \geq \int_0^\infty e^{-\lambda t}\left(e^{-\frac{2\alpha}{\mu_2}t-\frac{2\,\alpha}{3\mu_2^2}\mu_3+1}\right)^k dt=$

$$e^{k\left(1-\frac{2\,\alpha}{3\mu_2^2}\mu_3\right)}\int_0^\infty e^{-\left(\lambda+k\frac{2\alpha}{\mu_2}\right)t}dt = e^{k\left(1-\frac{2\,\alpha}{3\mu_2^2}\mu_3\right)}\frac{-1}{\lambda+k\frac{2\alpha}{\mu_2}}\left[e^{-\left(\lambda+k\frac{2\alpha}{\mu_2}\right)t}\right]_0^\infty$$

$$= e^{k\left(1-\frac{2\,\alpha}{3\mu_2^2}\mu_3\right)}.\frac{\mu_2}{\mu_2\lambda+2k\alpha}.\square$$

Proposition 2.11

If the service time distribution is *DFR*[3]

$$m_k \geq e^{k\left(\frac{1-\gamma_S^2}{2}\right)}\frac{\alpha}{k+\rho}, k = 1,2,\ldots \qquad (2.13).$$

Dem:

If the service is *DFR*$1-G(x)\geq e^{-\frac{x}{\alpha}-\frac{\gamma_S^2}{2}+\frac{1}{2}}$.

So,

$$m_k \geq \frac{1}{\alpha^k}\int_0^\infty e^{-\lambda t}\left[\int_t^\infty e^{-\frac{x}{\alpha}-\frac{\gamma_S^2}{2}+\frac{1}{2}}dx\right]^k dt = \frac{e^{k\left(\frac{1-\gamma_S^2}{2}\right)}}{\alpha^k}\int_0^\infty e^{-\lambda t}\left[\int_t^\infty e^{-\frac{x}{\alpha}}dx\right]^k =$$

$$e^{k\left(\frac{1-\gamma_S^2}{2}\right)}\frac{\alpha}{k+\rho}.\square$$

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[2] $G^*(x)=\frac{1}{\alpha}\int_0^x[1-G(y)]dy$ is the service time equilibrium distribution.

[3] For more details about *NBUE* (New Better than Used in Expectation), *NWUE* (New Worse than Used in Expectation), *IMRL* (Increasing Mean Residual Life) and *DFR* (Decreasing Failure Rate) distributions, important in reliability theory, see (4).

Proposition 2.12

If the service time distribution has d. f. given by, see (10):

$$G(x)=1-\frac{1}{\lambda}\frac{(1-e^{-\rho})e^{-\lambda x-\int_0^x\beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w\beta(u)du}dw-(1-e^{-\rho})\int_0^x e^{-\lambda w-\int_0^w\beta(u)du}dw},x\geq 0,-\lambda\leq\frac{\int_0^x\beta(u)du}{x}\leq\frac{\lambda}{e^{\rho}-1}\quad(2.14),$$

$$m_k=\int_0^\infty e^{-\lambda t}\left[1+\frac{1}{\rho}\ln\left[1-\frac{(1-e^{-\rho})\int_0^t e^{-\lambda w-\int_0^w\beta(u)du}du}{\int_0^\infty e^{-\lambda w-\int_0^w\beta(u)du}dw}\right]\right]^k dt,k=0,1,\ldots\qquad(2.15)$$

Dem:

Just substitute (2.14) into (2.1). □

Note: It is not known an expression to the sojourn time value in state k for the $M|G|\infty$ queue systems, apart from

a) $k=0$, for every $G(.)$, being

$$m_0=\frac{1}{\lambda}\qquad(2.16)$$

b) Every k, *for* exponential service time, where

$$m_k=\frac{\alpha}{k+\rho},k=0,1,\ldots\quad(2.17).$$

In the same circumstances, the Markov renewal process supplies the same results: indeed (2.16) is equal to (2.2) and if $G(x)=1-e^{-\frac{x}{\alpha}},x\geq 0$ in (2.1) it is obtained (2.17).

-The bounds given by (2.10), (2.11), match the exact value given by (2.17). The expression (2.13) is coincident with (2.17) for $\gamma_s=1$.

3. State 0 Recurrence Mean Time

For the Markov renewal process, the state 0 mean recurrence time[4] is given by:

$$\mu_0=\frac{1}{\lambda}\left[1+\sum_{j=1}^{\infty}\prod_{k=1}^{j}\frac{\lambda m_k}{1-\lambda m_k}\right]\qquad(3.1).$$

Proposition 3.1

If $\rho\leq\frac{1}{\gamma_s^2+1}$, $\mu_0\leq\frac{e^{\rho(\gamma_s^2+1)}}{\lambda}\qquad(3.2)$

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[4] Indeed, is the $M|G|\infty$ queue busy cycle mean time, see (5).

Dem: To use an upper bound of m_k in (3.1) it is necessary to certify that it is lesser than $\frac{1}{\lambda}$. The condition $\rho(\gamma_s^2+1)\leq 1$, due to Proposition 2.6, guaranties that E_2fulfills that request for $k\geq 1$.

So, $$\mu_0\leq\frac{1}{\lambda}\left[1+\sum_{j=1}^{\infty}\prod_{k=1}^{j}\frac{\frac{\rho(\gamma_s^2+1)}{k+1}}{1-\frac{\rho(\gamma_s^2+1)}{k+1}}\right]=\frac{1}{\lambda}\left[1+\sum_{j=1}^{\infty}\prod_{k=1}^{j}\frac{\rho(\gamma_s^2+1)}{k+1-\rho(\gamma_s^2+1)}\right]\leq \frac{1}{\lambda}\left[1+\sum_{j=1}^{\infty}\frac{[\rho(\gamma_s^2+1)]^j}{j!}\right]=\frac{e^{\rho(\gamma_s^2+1)}}{\lambda}\ .\square$$

Obs: For the $M|G|\infty$ queue systems

$\mu_0=\frac{e^{\rho}}{\lambda}$ (3.3).

So, in these conditions, the relative error arising from considering (3.2) instead of (3.1) is:

$$\frac{\frac{e^{\rho(\gamma_s^2+1)}}{\lambda}-\frac{e^{\rho}}{\lambda}}{\frac{e^{\rho}}{\lambda}}=e^{\rho\gamma_s^2}-1\leq e^{\frac{\gamma_s^2}{\gamma_s^2+1}}-1<e-1.$$

But $e^{\frac{\gamma_s^2}{\gamma_s^2+1}}-1\leq r\Leftrightarrow\frac{\gamma_s^2}{\gamma_s^2+1}\leq\log(r+1)\Leftrightarrow\gamma_s^2\leq\frac{\log(r+1)}{1-\log(r+1)}$.

That is: if $\rho(\gamma_s^2+1)\leq 1$, the relative error arising from taking the bound given by (3.2) instead of the true value given by (3.3) for μ_0 is such that:

a)$\varepsilon\leq e^{\frac{\gamma_s^2}{\gamma_s^2+1}}-1$,

b)$\varepsilon=0$ if$\gamma_s^2=0$,

c)$\varepsilon<e-1$,

d)$\varepsilon\leq r\ \ (r<e-1)$since$\gamma_s^2\leq\frac{\log(r+1)}{1-\log(r+1)}$.

So, requesting that ε is lesser than a given r, it results a criterion to measure the goodness of the m_k approximation by E_2 for a certain γ_s^2, since $\rho(\gamma_s^2+1)\leq 1$.

Being B the $M|G|\infty$queue busy period length, see (5-6),

$$E[B]=\frac{e^{\rho}-1}{\lambda}\,(3.4).$$

For the Markov renewal process, since $\rho(\gamma_s^2+1)\leq 1$,

$E[B]=\frac{e^{\rho(\gamma_s^2+1)}-1}{\lambda}$ (3.5).

Now, the relative error owns to take (3.5) instead (3.4), is

$$\frac{\frac{e^{\rho(\gamma_s^2+1)}-1}{\lambda}-\frac{e^{\rho}-1}{\lambda}}{\frac{e^{\rho}-1}{\lambda}}=\frac{e^{\rho(\gamma_s^2+1)}-e^{\rho}}{e^{\rho}-1}=\frac{e^{\rho\gamma_s^2}-1}{1-e^{-\rho}}\leq\frac{e^{\frac{\gamma_s^2}{\gamma_s^2+1}}-1}{1-e^{-\rho}}<\frac{e-1}{1-e^{-\rho}}.$$ So

a) $\delta \leq \frac{e^{\frac{\gamma_s^2}{\gamma_s^2+1}}-1}{1-e^{-\rho}}$,

b) $\varepsilon = 0$ if $\gamma_s^2 = 0$,

c) $\delta < \frac{e-1}{1-e^{-\rho}}$,

d) $\delta \leq r\left(r < \frac{e-1}{1-e^{-\rho}}\right)$ since $\gamma_s^2 \leq \frac{\log(r(1-e^{-\rho})+1)}{1-\log(r(1-e^{-\rho})+1)}$.

So, the bounds for δ are greater than the obtained to ε. Then it is preferable to use a criterion based on ε than on δ, to measure the goodness of the approximation of m_k by E_2.

4. Mean Number of Entries in State *k* between Two

Entries in State 0

For the Markov renewal process, the mean number of entries in state *k* between two entries in state 0 is:

$$v_k = \lambda^{k-1}\frac{m_1 \dots m_k}{(1-m_1)\dots(1-m_k)}, k = 1,2,\dots \qquad (4.1).$$

Proposition 4.1

If $\rho \leq \frac{1}{\gamma_s^2+1}$

$$v_k \leq (k+1)\frac{\rho^{k-1}\left(\gamma_s^2+1\right)^{k-1}}{k!}, k = 1,2,\dots \qquad (4.2).\square$$

Obs.: Values for $v_k, k = 1,2,\dots$ for the $M|G|\infty$ queue system are not known,

From (2.2), (2.9) and (4.2) it follows:

$$m_k v_k \leq \frac{\alpha\rho^{k-1}\left(\gamma_s^2+1\right)^{k}}{k!}, k = 0,1,\dots \quad (4.3)$$

Since $\rho(\gamma_s^2 + 1) \leq 1$.

For the $M|G|\infty$ queue system

$m_k v_k = \frac{\alpha \rho^{k-1}}{k!}, k = 0,1, \dots$ (4.4).

But $\frac{\frac{\alpha\rho^{k-1}(\gamma_s^2+1)^k}{k!} - \frac{\alpha\rho^{k-1}}{k!}}{\frac{\alpha\rho^{k-1}}{k!}}$ $=(\gamma_s^2 + 1)^k - 1$, that is null for $\gamma_s = 0$ or $k = 0$ and increases with $k$if $\gamma_s^2 > 0$.

Note: For $\rho \leq 1\ and\ \gamma_s^2 = 0$ the Markov renewal process supplies the following results:

a) $m_0 = \frac{1}{\lambda}$,

b) $m_k \leq \frac{\alpha}{k+1}, k = 1,2, \dots$

c) $\mu_0 \leq \frac{e^\rho}{\lambda}$,

d) $E[B] \leq \frac{e^\rho - 1}{\lambda}$,

e) $v_k \leq (k + 1)\frac{\rho^{k-1}}{k!}\ , k = 1,2, \dots$

f) $m_k v_k = \frac{\alpha\rho^{k-1}}{k!}, k = 0,1, \dots$

So, the value obtained for m_0 coincides with the $M|G|\infty$ one. And the bounds obtained for μ_0, $E[B]$ and $m_k v_k$ coincide with the true values for the same $M|G|\infty$quantities.But the bounds obtained for m_k and $v_k$coincide with the true value obtained when the service time distribution is exponential and the traffic intensity is 1. In opposition, the bound got for m_k cannot coincide with the one given by (2.15) for $\rho < 1$. So, it is excluded the hypothesis of having an expression for m_k independent from the service time distribution and equal to the one given by (2.15). Then, only rarely the Markov renewal process gives values for μ_0, $E[B]$ and $m_k v_k$ identical to the $M|G|\infty$ ones.

If $\rho(\gamma_s^2 + 1) \leq 1$ it is possible, after the Markov renewal process, to get upper bounds for the $M|G|\infty$system quantities μ_0, $E[B]$ and $m_k v_k$. So, it is admissible to consider that at least E_2, beyond being a m_k upper bound for the Markov renewal process, also plays the same role for the $M|G|\infty$ queue.

Note still that if $\gamma_s^2 = 0$, for the Markov renewal process:

$$m_k = \int_0^\alpha e^{-\lambda t}\left[1 - \frac{t}{\alpha}\right]^k dt,\ \ k = 0,1 \dots \qquad (4.5).$$

So, for $k \geq 1$, $m_k \leq \int_0^\alpha \left(1 - \frac{t}{\alpha}\right)^k dt$=$\left[\frac{-\alpha}{k+1}\left(1 - \frac{t}{\alpha}\right)^{k+1}\right]_0^\alpha$, That is $m_k \leq \frac{\alpha}{k+1}, k = 1,2, \dots$.

But, requesting that $\frac{\alpha}{k+1} \leq \frac{1}{\lambda} \Leftrightarrow k \geq \rho - 1$ that leads to $\rho - 1 \leq 0 \Leftrightarrow \rho \leq 1$.

-$m_1 = \int_0^\alpha e^{-\lambda t}\left(1 - \frac{t}{\alpha}\right)\ dt = \left[-\frac{e^{-\lambda t}}{\lambda}\left(1 - \frac{t}{\alpha}\right)\right]_0^\alpha -$ $\int_0^\alpha -\frac{e^{-\lambda t}}{\lambda}\left(-\frac{1}{\alpha}\right) dt = \frac{1}{\lambda} - \frac{1}{\rho}\int_0^\alpha e^{-\lambda t} dt = \frac{1}{\lambda} - \frac{1}{\rho}\left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^\alpha = \frac{1}{\lambda} - \frac{1}{\rho}\left(-\frac{e^{-\rho}}{\lambda} + \frac{1}{\lambda}\right)$. So,

$m_1 = \alpha \frac{\rho + e^{-\rho} - 1}{\rho^2}$ (4.6).

And, integrating by parts,

$m_{k+1} = \frac{1}{\lambda} - \frac{k+1}{\rho} m_k$, k=1, 2… (4.7).

With (4.6) and (4.7) it is possible to obtain $m_k,\ k = 1,2, \dots$ for $\gamma_s^2 = 0$ and it is possible to conclude that, in this case, (2.17) does not hold.

Proposition 4.2

If the service time distribution is *NBUE*

a)$\mu_0 \leq \frac{e^\rho}{\lambda}$,

b)$E[B] \leq \frac{e^\rho - 1}{\lambda}$,

c)$\upsilon_k \leq (k+1)\frac{\rho^{k-1}}{k!}$, $k = 1,2, \dots$

d) $m_k \upsilon_k \leq \frac{\alpha \rho^{k-1}}{k!}, k = 0,1, \dots$

Obs: The bounds obtained for μ_0, $E[B]$ and $m_k \upsilon_k$ coincide with the true value of these quantities for the $M|G|\infty$ queue.

If the service time distribution is *NWUE*

a)$\mu_0 \geq \frac{e^\rho}{\lambda}$,

b)$E[B] \geq \frac{e^\rho - 1}{\lambda}$,

c)$\upsilon_k \geq (k+1)\frac{\rho^{k-1}}{k!}$, $k = 1,2, \dots$

d) $m_k \upsilon_k \geq \frac{\alpha\rho^{k-1}}{k!}, k = 0,1, \ldots$

With a comment identical to the one in the case *NBUE*.

So, it is admissible that $\frac{\alpha}{k+\rho}, k = 0,1, \ldots$ is an upper bound (lower bound) for the true value of $m_k$in the $M|G|\infty$ queue systems in the case of *NBUE* (*NWUE*) service time distributions.

After (2.1) and integrating by parts
$m_{k+1} = \int_0^\infty e^{-\lambda t}\left[\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right]^{k+1} dt = \left[-\frac{e^{-\lambda t}}{\lambda}\left(\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right)^{k+1}\right]_0^\infty -$
$\int_0^\infty -\frac{e^{-\lambda t}}{\lambda}(k+1)\left[\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right]^{k} \frac{G(t)-1}{\alpha} dt = \frac{1}{\lambda} - \frac{k+1}{\rho}\int_0^\infty e^{-\lambda t}\left[\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right]^{k} \big(1 -$
$G(t)\big)dt \geq \frac{1}{\lambda} - \frac{k+1}{\rho} m_k$. So

$$m_{k+1} \geq \frac{1}{\lambda} - \frac{k+1}{\rho} m_k\,, \quad k = 1,2, \ldots (4.8).$$

Note: According to (4.7), when the service is constant, the equality holds in (4.8).

5. Sojourn Time in State k Distribution

The sojourn time in sate k distribution function for the Markov renewal process is:

$$C_k(t) = 1 - e^{-\lambda t}\left[\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right]^{k}, t \geq 0, k = 0,1, \ldots \quad (5.1).$$

Evidently,

Proposition 5.1

$$C_k(t) \geq 1 - e^{-\lambda t}, t \geq 0, k = 0,1, \ldots \quad (5.2).$$

Proposition 5.2

If the service time distribution is exponential

$$C_k(t) = 1 - e^{-\frac{1}{\alpha}(k+\rho)t}, t \geq 0, k = 0,1, \ldots \quad (5.3).$$

Obs.: This result is coincident with the one known for the $M|G|\infty$ queue.

Proposition 5.3

$$C_0(t) = 1 - e^{-\lambda t}, t \geq 0, \qquad (5.4).$$

Obs.: Result obvious for any $M|G|\infty$ queue and for any queue with Poisson arrivals process.

Proposition 5.4

If the service time distribution is *NBUE*

$$C_k(t) \geq 1 - e^{-\frac{1}{\alpha}(k+\rho)t}, t \geq 0, k = 0,1, \dots \quad (5.5).$$

Obs.: As emphasized before, (5.5), beyond supplying a lower bound for $C_k(t)$ in the Markov renewal process, also gives a lower bound for that quantity in the $M|G|\infty$ system for the case of *NBUE* service time.

If the service time distribution is *NWUE*

$$C_k(t) \leq 1 - e^{-\frac{1}{\alpha}(k+\rho)t}, t \geq 0, k = 0,1, \dots \quad (5.6)$$

And it is pertinent a comment analogous to the former one with the change of lower bound by upper bound.

Proposition 5.5

If the service time distribution is *IMRL*

$$C_k(t) \leq 1 - e^{-\lambda t + k\left(-\frac{2\alpha}{\mu_2}t - \frac{2\ \alpha}{3\mu_2^2}\mu_3 + 1\right)}, t \geq 0, k = 0,1, \dots \quad (5.6)$$

Proposition 5.6

If the service time distribution is *DFR*

$$C_k(t) \leq 1 - e^{-\frac{1}{\alpha}(k+\rho)t + k\frac{1-\gamma_S^2}{2}}, t \geq 0, k = 0,1, \dots \quad (5.7).$$

Proposition 5.7

For the service time given by (2.14),

$$C_k(t) = 1 - e^{-\lambda t}\left[1 + \frac{1}{\rho} ln\left[1 - \frac{\left(1 - e^{-\rho}\right)\int_0^t e^{-\lambda w - \int_0^w \beta(u)du}\, du}{\int_0^\infty e^{-\lambda w - \int_0^w \beta(u)du}\, dw}\right]\right]^k, t \geq 0, k = 0,1, \dots \quad (5.8).$$

Conclusions

When analytical exact results are not available, numerical methods are used to try to find approximations for the interesting quantities under study. It is what is done in this work for the $M|G|\infty$queue, trying to approximate it for a Markov renewal process. An alternative is using simulation methods. For this approach see, for instance (6, 7).

Still another is to determine service time distributions for which it is possible to determine most of the interesting quantities for the $M|G|\infty$queue. This is made solving differential equations induced for the study of the transient behavior, see (8-10).

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