1/f Noise: Branching Process Model (I); Formalization

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Abstract : In order to discuss the $1/f$ problem, the statistics of branching process of particle in a multiplicative medium are developed by taking account of random particle immigration. The probability distribution function of the number of particles founded in the medium at any fixed time and/or of particles detected in a time interval are given in closed form. Also given are conditional probabilities, conditioned on having a fixed number of particles at time t=0. These are applied to the case that exactly two particles are produced by branching.

1. Introduction

Since the observation of the $1/f$ power spectral density (PSD) of shot noise by Johnson in 1925[(1)], many works have been published to elucidate the phenomenon theoretically, where f represents the frequency of the phenomenon. Most of them, however, are insufficient to simulate the phenomenon with a $1/f$ PSD covering a wide range of frequencies or are too sophisticated. The aim of the present work is to give a model for generating a series in which its PSD is characterized by a $1/f$ distribution for a wide range of frequencies.

When a PSD of events $x(t)$ at time t behaves like $1/f^a$ $(a \geq 0)$, a large exponent a results in a long-term correlation between the events because the intensities of fluctuations with low frequencies are relatively large compared with those with high frequencies. In the case that $0 \leq a < 1$, the correlation function is given by

$$\langle x(t)x(t+\tau)\rangle = \begin{cases} \tau^{a-1} & (0 < a < 1) \\ \delta(\tau) & (a = 0) \end{cases} \tag{1-1}$$

which is independent of time t, and the phenomenon is stationary. When $a > 1$, the correlation function depends on t and the non-stationary phenomenon occurs. Therefore, phenomena with a $1/f$ PSD are intermediate between stationary and non-stationary phenomena.

The basic idea of the present work is to give the representations for generating the series with a PSD behaving like $1/f$ by using the branching process model. Here an event may have correlation with other events through the branching processes. The branching process model was first applied in 1874 to discuss the statistics of family lines[(2)]. Many other applications of this model have been made in various fields, i.e. to the discussion of the fluctuations of the neutron number in a nuclear reactor, to that of electron number in solids, to the discussion of the particle spectra obtained by a high-energy collider, and so on.

In the model of this work, the statistics of particle number in a medium, in which a particle may branch into several particles, be captured or be observed by a detector in the presence of random particle immigration, are developed and the representations of time series are derived. A part of the present discussions is already developed in my previous works[3, 4].

2. Branching started from a single particle

Consider a medium in which a particle may be subjected to capture and multiplicative reactions. By a multiplicative reaction, several particles appear in the medium and a branch is produced. We suppose in the present work that the system is homogeneous, i.e. the system is well stirred or the velocity of particles is infinitely large.

2.1. Existing particles

Suppose no particle existed in the medium before the time $t = 0$, and one particle has been injected at $t = 0$. We consider then the probability $p(n,t)$ that n particles are found in the medium at time $t > 0$. When δt is an infinitely short time interval, the probability $p(n, t + \delta t)$ that n particles are found in the medium at time $t + \delta t$ is given by the sum of the probability that n particles are found at t and nothing happens during the successive time interval δt, the probability that $n + 1$ particles are found at t and one particle is absorbed during the successive δt, and the probability that $n - \nu + 1$ particles are found at t and one particle branches out into ν particles during the successive δt, i.e.

$$\begin{aligned} p(n, t + \delta t) &= p(n,t)(1 - n\lambda_t \delta t) + p(n+1,t)(n+1)\lambda_a \delta t \\ &\quad + \sum_{\nu=0}^{n} p(n-\nu+1,t)(n-\nu+1)\lambda_m p_\nu \delta t\,, \end{aligned} \tag{2-1}$$

where λ_a and λ_m are probabilities of absorption and multiplicative reactions, respectively, in unit time for a particle,

$$\lambda_t = \lambda_a + \lambda_m, \tag{2-2}$$

and p_ν is the distribution function of the number of particles produced by a multiplicative reaction. It is normalized to $\sum_{\nu=2}^{\infty} p_\nu = 1$. Introducing the probability generation functions (PGF)

$$h(w,t) = \sum_{n=0}^{\infty} w^n p(n,t) \tag{2-3}$$

and

$$\Psi(w) = \sum_{\nu=2}^{\infty} w^\nu p_\nu, \tag{2-4}$$

Eq. (2-1) can be expressed as

$$\begin{aligned} \frac{\partial h(w,t)}{\partial t} &= \{\lambda_m \Psi(w) - \lambda_t w + \lambda_a\} \frac{\partial h(w,t)}{\partial w} \\ &= \Phi(w) \frac{\partial h(w,t)}{\partial w}\quad, \end{aligned} \tag{2-5}$$

where

$$\Phi(w) = \lambda_m \Psi(w) - \lambda_t w + \lambda_a. \tag{2-6}$$

The initial and boundary conditions for $h(w,t)$ are

$$h(w, 0) = w \tag{2-7}$$

and

$$h(1, t) = 1, \tag{2-8}$$

respectively. When λ_a and λ_m are constant, the general solution of the linear partial differential equation Eq.(2-5) is solved in closed form so that

$$h(w, t) = F\{t + k(w)\}, \tag{2-9}$$

where

$$k(w) = \int \frac{dw}{\Phi(w)} \tag{2-10}$$

and $F(w)$ is the inverse function of $k(w)$, i.e. $F[k(w)] = w$ due to Eq.(2-7). Equation (2-8) is satisfied since $k(1)=\infty$ and so $F(\infty)=1$.

The probability $p(n, t)$ is calculated by the following equation,

$$p(n, t) = \frac{1}{n!}\left(\frac{\partial^n h}{\partial w^n}\right)_{w=0}. \tag{2-11}$$

2.2. Counted particles

In order to consider the statistics of particles counted by a detector in the medium, we introduce the probability $p(m, n, t)$ that the detector counts m particles during a time interval (0, t) and n particles are found in the medium at time $t > 0$ when one particle has been injected at $t = 0$. Introducing the PGF of $p(m, n, t)$,

$$g(v, w, t) = \sum_{m=0}^{\infty} v^m \sum_{n=0}^{\infty} w^n p(m, n, t). \tag{2-12}$$

the equation equivalent to Eq.(2-5) is represented to

$$\frac{\partial g(v, w, t)}{\partial t} = \Psi(v, w)\frac{\partial g(v, w, t)}{\partial w}. \tag{2-13}$$

The function $\Psi(v, \omega)$ in Eq.(2-13) is given by

$$\Psi(v, w) = \lambda_m \Phi(w) - (\lambda_m + \lambda_c + \lambda_d)w + \lambda_c + \lambda_d v \tag{2-14}$$

when the detector is of absorption type, that is, the counted particle is absorbed by the detector. In Eq.(2-14), λ_d is the detection rate for a particle, and λ_c describes capture of particles in the medium other than absorption by the detector, and, therefore, the absorption rate λ_a is described by

$$\lambda_a = \lambda_c + \lambda_d \tag{2-15}$$

in this case. When the detector is of non-absorption type, the function $\Psi(v, \omega)$ is expressed by

$$\Psi(v, w) = \lambda_m \Phi(w) - (\lambda_m + \lambda_c + (1 - v)\lambda_d)w + \lambda_c, \tag{2-16}$$

and λ_a in this case is equivalent to λ_c, i.e.

$$\lambda_a = \lambda_c. \tag{2-17}$$

The PGF $g(v, w, t)$ satisfies the conditions

$$g(v, w, 0) = w,. \tag{2-18}$$

$$g(1,1, t) = 1 \tag{2-19}$$

and

$$g(1,w,t) = h(w,t). \tag{2-20}$$

The solution of Eq.(2-13) is

$$g(v,w,t) = I\big(v, t + J(v,w)\big), \tag{2-21}$$

where

$$J(v,\omega) = \int \frac{dw}{\Psi(v,w)} \tag{2-22}$$

and $I(v,w)$ is the inverse function of $J(v,w)$ with regard to w, i.e. $I\big(v, J(v,w)\big) = w$.

3. Branching with random immigration

When a random source exists in the medium, all the individual particles from the source can produce branches and, therefore, many branches having different origins are found in the medium.

3.1. Existing particles

Suppose $P_k(n,t)$ is the probability that n particles are found in the medium at time $t > 0$ after we had k particles at $t = 0$.. The PGF of $P_k(n,t)$,

$$G(u,w,t) = \sum_{k=0}^{\infty} u^k \sum_{n=0}^{\infty} w^n P_k(n,t), \tag{3-1}$$

is given by

$$G(u,w,t) = \frac{1}{1 - uh(\mathrm{w},\mathrm{t})} Q(w,t). \tag{3-2}$$

Here

$$Q(w,t) = \exp\left\{\int_0^t S(\tau)[h(w,t-\tau) - 1]d\tau\right\}, \tag{3-3}$$

and $S(t)$ is the random immigration rate of source particles at time t. The coefficient of Eq.(3-2), $Q(w,t)$, gives the effect of source particles appearing in $(0, t)$ and the other part of $G(u,w,t)$ is that of particles existing at $t = 0$. The formation of Eq.(3-2) is described in detail in Ref.(3).

The conditional probability $P_k(n,t)$ can be obtained easily and expressed as

$$\begin{aligned} P_k(n,t) &= \frac{1}{k!\,n!}\left(\frac{\partial^k}{\partial u^k}\frac{\partial^n}{\partial w^n} G(u,w,t)\right)_{u=w=0}, \\ &= \sum_{i=0}^{n} K_k^{(n-i)} Q_0^{(i)} \end{aligned} \tag{3-4}$$

where

$$Q_0^{(i)} = \frac{1}{i!}\left(Q^{(i)}\right)_{w=0} = \frac{1}{i!}\left(\frac{\partial^i}{\partial w^i} Q(w,t)\right)_{w=0} \tag{3-5}$$

and

$$K_k^{(n-i)} = \sum_{l=0}^{n-i} p(l,t) K_{k-1}^{(n-i-l)}, \tag{3-6}$$

which is obtained successively using the relation

$$K_0^{(i)} = \delta_{i,0}. \tag{3-7}$$

The term $Q_0^{(i)}$ will be calculated when a more definite model is given.

3.2. Counted particles

We next consider the conditional probability $P_k(m,n,t)$ that *m* counts have been recorded during a time interval (0, *t*) and *n* particles are found in the medium at time $t > 0$ after we had *k* particles at $t = 0$. Repeating the familiar procedure, we obtain the expression for the PGF of $P_k(m,n,t)$,

$$T(u,v,w,t) = \sum_{k=0}^{\infty} u^k \sum_{m=0}^{\infty} v^m \sum_{n=0}^{\infty} w^n P_k(m,n,t), \tag{3-8}$$

as

$$T(u,v,w,t) = \frac{1}{1 - ug(v,\mathrm{w},\mathrm{t})} R(v,w,t), \tag{3-9}$$

where

$$R(v,w,t) = \exp\left\{\int_0^t S(\tau)[g(v,w,t-\tau) - 1]d\tau\right\}. \tag{3-10}$$

The coefficient $R(v,w,t)$ in Eq.(3-9) is the effect of source particles appearing in (0, *t*) and the other part of $T(u,v,w,t)$ is that of particles existing at $t = 0$.

The conditional probability $P_k(m,n,t)$ is obtained similarly as before;

$$P_k(m,n,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} K_k^{(m-i,n-j)} R_0^{(i,j)}, \tag{3-11}$$

where

$$R_0^{(i,j)} = \frac{1}{i!\,j!}\left(R^{(i,j)}\right)_{v=w=0} = \frac{1}{i!\,j!}\left(\frac{\partial^i}{\partial v^i}\frac{\partial^j}{\partial w^j} R(v,w,t)\right)_{v=w=0} \tag{3-12}$$

and

$$K_k^{(m-i,n-j)} = \sum_{l_1=0}^{m-i} \sum_{l_2=0}^{n-j} p(l_1,l_2,t) K_{k-1}^{(m-i-l_1,n-j-l_2)}. \tag{3-13}$$

The coefficient $K_k^{(m-i,n-j)}$ for $k > 0$ can be calculated successively using the relation

$$K_0^{(i,j)} = \delta_{i,0}\delta_{j,0}. \tag{3-14}$$

4. Application

In the above formation, the branching is described by a distribution function p_ν. Usually, p_ν is unknown. In the present section, we suppose that exactly two particles are produced by a multiplicative reaction, which is a particular but very important and useful branching mode. The distribution function p_ν is, therefore,

$$p_\nu = \begin{cases} 1 & (\nu = 2) \\ 0 & (\nu \neq 2) \end{cases} \tag{4-1}$$

and its PGF defined by Eq.(2-4) is given by

$$\Psi(w) = w^2. \tag{4-2}$$

4.1. Branching started from a single particle

4.1.1. Existing particles

Equation (2-6) is written as

$$\Phi(w) = \lambda_a(w-1)(\mu w - 1), \tag{4-3}$$

where

$$\mu = \frac{\lambda_m}{\lambda_a}. \tag{4-4}$$

This ratio can be considered a multiplication rate of a particle. In Eq.(4-4), λ_a is given by Eq.(2-15) and (2-17) according to the detector type.

(i) Case of $\mu \neq 1$ (non-critical case)

The function $k(\omega)$ defined by Eq.(2-10) is, in this case,

$$k(w) = \frac{1}{\alpha}\ln\left|\frac{w - {}^1\!/_\mu}{w-1}\right|, \tag{4-5}$$

where

$$\alpha = \lambda_a - \lambda_m = (1-\mu)\lambda_a. \tag{4-6}$$

Consequently the PGF shown in Eq.(2-9) is, therefore, obtained as

$$h(w,t) = \frac{(\mu - e^{-\alpha t})w - (1 - e^{-\alpha t})}{\mu(1 - e^{-\alpha t})w - (1 - \mu e^{-\alpha t})}, \tag{4-7}$$

which satisfies the conditions (2-7) and (2-8). The probability $p(n,t)$ is given, using Eq.(2-11), by

$$p(n,t) = \begin{cases} W_1 & (n=0) \\ \left(\dfrac{1-\mu}{1-\mu e^{-\alpha t}}\right)^2 e^{-\alpha t} & (n=1), \\ \mu W_1 \cdot p(n-1,t) & (n \geq 2) \end{cases} \tag{4-8}$$

where

$$W_1 = \frac{1 - e^{-\alpha t}}{1 - \mu e^{-\alpha t}}. \tag{4-9}$$

(ii) Case of $\mu = 1$ (critical case)

In a similar way to the previous case,

$$h(w,t) = \frac{(\lambda_a t - 1)w - \lambda_a t}{\lambda_a t w - (\lambda_a t + 1)}. \tag{4-10}$$

This expression satisfies the condition (2-7) and (2-8) as well. The probability $p(n,t)$ in this case is described as

$$p(n,t) = \begin{cases} W_2 & (n=0) \\ \left(\frac{W_2}{\lambda_a t}\right)^2 & (n=1), \\ W_2 \cdot p(n-1,t) & (n \geq 2) \end{cases} \tag{4-11}$$

where

$$W_2 = \frac{\lambda_a t}{\lambda_a t + 1}. \tag{4-12}$$

4.1.2. Counted particles

Using the function $\Psi(v,w)$ which is obtained by substituting Eq.(2-6) with Eq.(4-2) for Eq.(2-14) or Eq.(2-16), the PGF in Eq.(2-21) is derived by the familiar procedure

$$g(v,w,t) = \frac{\left(\xi - \eta e^{-\theta t}\right)w - \eta\xi\left(1 - e^{-\theta t)}\right)}{(1 - e^{-\theta t)})w - (\eta - \xi e^{-\theta t})}, \tag{4-13}$$

where, for an absorption-type detector,

$$\begin{aligned} \theta &= \sqrt{(\lambda_c + \lambda_d - \lambda_m)^2 + 4\lambda_d\lambda_m(1-v)} \\ \eta &= \frac{\lambda_t + \theta}{2\lambda_m} \\ \xi &= \frac{\lambda_t - \theta}{2\lambda_m}, \end{aligned} \tag{4-14}$$

and, for a non-absorption-type detector,

$$\begin{aligned} \theta &= \sqrt{(\lambda_t - \lambda_d v)^2 + 4\lambda_c\lambda_m} \\ \eta &= \frac{\lambda_t - \lambda_d v + \theta}{2\lambda_m} \\ \xi &= \frac{\lambda_t - \lambda_d v - \theta}{2\lambda_m}. \end{aligned} \tag{4-15}$$

Here, in Eq.(4-14) and Eq.(4-15),

$$\lambda_t = \lambda_a + \lambda_m = \begin{cases} \lambda_c + \lambda_d + \lambda_m & \text{(absorption type)} \\ \lambda_c + \lambda_m & \text{(non - absorption type)} \end{cases}. \tag{4-16}$$

The probability is estimated from the following equation:

$$p(m,n,t) = \begin{cases} \frac{1}{m!}\left[\frac{\partial^m}{\partial v^m}\left\{\frac{\eta\xi(1-e^{-\theta t})}{\eta - \xi e^{-\theta t}}\right\}\right]_{v=0} & (n=0) \\ \frac{1}{m!}\left[\frac{\partial^m}{\partial v^m}\left\{(\eta-\xi)^2 e^{-\theta t}\frac{(1-e^{-\theta t})^{n-1}}{(\eta - \xi e^{-\theta t})^{n+1}}\right\}\right]_{v=0} & (n \geq 1) \end{cases}. \tag{4-17}$$

We cannot find the general form of the *m*th differential coefficient at v=0, but there will be no difficulty for numerical estimation of the probability, because the detection efficiency, λ_d, is usually small and estimation of the probability is, therefore, necessary only for small *m*.

The probability for *m*=0, $p(0,n,t)$, is easily estimated by

$$p(0,n,t)=\begin{cases}\eta_0\xi_0 V & (n=0)\\ \dfrac{(\eta_0-\xi_0)^2 e^{-\theta_0 t}}{(\eta_0-\xi_0 e^{-\theta_0 t})^2} & (n=1),\\ V\cdot p(0,n-1,t) & (n\geq 2)\end{cases} \tag{4-18}$$

where

$$V=\frac{1-e^{-\theta_0 t}}{\eta_0-\xi_0 e^{-\theta_0 t}}, \tag{4-19}$$

$$\eta_0=\frac{\lambda_t+\theta_0}{2\lambda_m}, \qquad \xi_0=\frac{\lambda_t-\theta_0}{2\lambda_m}, \tag{4-20}$$

and

$$\theta_0=\begin{cases}\sqrt{(\lambda_c+\lambda_d-\lambda_m)^2+4\lambda_d\lambda_m} & (\text{absorption type})\\ \sqrt{{\lambda_t}^2-4\lambda_m\lambda_c} & (\text{non}-\text{absorption type})\end{cases}. \tag{4-21}$$

Introducing the ratio of the detection rate to the absorption rate ε of a particle, this ratio ε and the multiplication rate μ given in Eq.(4-4) are described as follows;

$$\mu=\frac{\lambda_m}{\lambda_a}=\begin{cases}\frac{\lambda_m}{\lambda_c+\lambda_d} & (\text{absorption type})\\ \frac{\lambda_m}{\lambda_c} & (\text{non}-\text{absorption type})\end{cases}. \tag{4-22}$$

$$\varepsilon=\frac{\lambda_d}{\lambda_a}=\begin{cases}\frac{\lambda_d}{\lambda_c+\lambda_d} & (\text{absorption type})\\ \frac{\lambda_d}{\lambda_c} & (\text{non}-\text{absorption type})\end{cases}. \tag{4-23}$$

The parameters in Eq.(4-20) and Eq.(4-21) are given again with μ and ε , which are

$$\theta_0=\begin{cases}\lambda_a\sqrt{(1-\mu)^2+4\varepsilon\mu} & (\text{absorption type})\\ \lambda_a\sqrt{(1+\varepsilon+\mu)^2-4\mu} & (\text{non}-\text{absorption type})\end{cases}, \tag{4-24}$$

$$\eta_0=\begin{cases}\dfrac{1}{2\mu}\left(1+\mu+\sqrt{(1-\mu)^2+4\varepsilon\mu}\right) & (\text{absorption type})\\ \dfrac{1}{2\mu}\left(1+\varepsilon+\mu+\sqrt{(1+\varepsilon+\mu)^2-4\mu}\right) & (\text{non}-\text{absorption type})\end{cases}, \tag{4-25}$$

and

$$\xi_0=\begin{cases}\dfrac{1}{2\mu}\left(1+\mu-\sqrt{(1-\mu)^2+4\varepsilon\mu}\right) & (\text{absorption type})\\ \dfrac{1}{2\mu}\left(1+\varepsilon+\mu-\sqrt{(1+\varepsilon+\mu)^2-4\mu}\right) & (\text{non}-\text{absorption type})\end{cases}. \tag{4-26}$$

4.2. Branching with random immigration

4.2.1. Existing particles

The source strength $S(t)$ is assumed to be a constant S.

(i) Case of $\mu\neq 1$ (non-critical case)

The PGF of $P_k(n,t)$, i.e. $G(u,\omega,t)$ in Eq.(3-2), is given by

$$G(u,w,t)=\frac{(1-w)\mu e^{-\alpha t}-(1-\mu w)}{(\mu-u)(1-w)e^{-\alpha t}-(1-u)(1-\mu w)} \times \exp\left[\frac{S}{\mu\lambda_a}\ln\frac{1-u}{1-\mu e^{-\alpha t}-(1-e^{-\alpha t})\mu w}\right] \tag{4-27}$$

in the present binary branching mode. In the probability $P_k(n,t)$ described in Eq.(3-4), the factor $Q_0^{(i)}$ is obtained, in the binary branching mode, as

$$Q_0^{(i)}=\begin{cases}\exp\left[\frac{S}{\mu\lambda_a}\ln\frac{1-\mu}{1-\mu e^{-\alpha t}}\right] & (i=0)\\ \frac{\frac{S}{\mu\lambda_a}+i-1}{i}\mu W_1\cdot Q_0^{(i-1)} & (i\geq 1)\end{cases}, \tag{4-28}$$

where W_1 is given in Eq.(4-9). The probability $p(l,t)$ in Eq.(3-6) is estimated with Eq.(4-8) in the binary mode.

(ii) Case of μ =1 (critical case)

In a similar way

$$G(u,w,t)=\frac{-\{1+\lambda_a t(1-w)\}}{uw-1-\lambda_a t(1-u)(1-w)} \times \exp\left[-\frac{S}{\lambda_a}\ln\{1+\lambda_a t(1-w)\}\right], \tag{4-29}$$

and

$$Q_0^{(i)}=\begin{cases}\exp\left[-\frac{S}{\lambda_a}\ln(1+\lambda_a t)\right] & (i=0)\\ \frac{\frac{S}{\lambda_a}+i-1}{i}\mu W_2 Q_0^{(i-1)} & (i\geq 1)\end{cases}. \tag{4-30}$$

Here W_2 is given in Eq.(4-12), and the probability $p(l,t)$ in Eq.(3-6) is estimated with Eq.(4-11).

4.2.2. Counted particles

In the present binary branching mode, the PGF in Eq.(3-9) and the coefficient $R(v,w,t)$ in Eq.(3-10) are easily obtained as

$$T(u,v,w,t) = \frac{\left[(1-e^{-\theta t})w-(\eta-\xi e^{-\theta t})\right]\times R(v,w,t)}{[(1-\xi u)-(1-\eta u)e^{-\theta t}]w-[\eta(1-\xi u)-\xi(1-\eta u)e^{-\theta t}]} \tag{4-31}$$

and

$$R(v,w,t)=\exp\left[(\xi-1)St+\frac{S}{\mu\lambda_a}\ln\frac{\eta-\xi}{(\eta-\xi e^{-\theta t})-(1-e^{-\theta t})w}\right] \tag{4-32}$$

In the coefficient $R_0^{(i,j)}$ in Eq.(3-12) for calculating the probability $P_k(m,n,t)$ in Eq.(3-11), the order i is the number of counts due to source particles appearing in a time interval (0, t). It is, therefore, sufficient to estimate $R_0^{(i,j)}$ only for small i when the detection efficiency is small. When i=0, this coefficient is described as

$$R_0^{(0,j)}=\frac{1}{j!}\left(\frac{\partial^j}{\partial w^j}R(v,w,t)\right)_{v=w=0}$$

$$=\begin{cases} \exp\left[(\xi_0 - 1)St + \frac{S}{\mu\lambda_a}\ln\frac{\eta_0-\xi_0}{\eta_0-\xi_0 e^{-\theta_0 t}}\right] & (j=0) \\ \frac{\frac{S}{\lambda\mu_a}+j-1}{j}\cdot V\cdot R_0^{(0,j-1)} & (j\geq 1) \end{cases}, \tag{4-33}$$

where

$$V = \frac{1-e^{-\theta_0 t}}{\eta_0-\xi_0 e^{-\theta_0 t}}. \tag{4-34}$$

The probability $P_k(0,n,t)$ is, therefore, expressed as

$$P_k(0,n,t) = \sum_{j=0}^{n} K_k^{(0,n-j)}\cdot R_0^{(0,j)}, \tag{4-35}$$

where

$$K_k^{(0,n-j)} = \sum_{l=0}^{n-j} p(0,l,t)K_{k-1}^{(0,n-j-l)} \tag{4-36}$$

$$K_0^{(0,j)} = \delta_{j,0}.$$

Using the representations obtained in this paper, a series formed by time intervals between successive detections of particles will be generated in a computer and be discussed the $1/f$ problem in the next paper.

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