On Generalized Convergence in Collatz Systems and $\delta k + \beta$ Dynamical Systems

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April 8, 2021

ON GENERALIZED CONVERGENCE IN COLLATZ SYSTEMS AND $\delta k+\beta$ DYNAMICAL SYSTEMS.

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Abstract

The Collatz Conjecture belongs to the generalized class of Dynamical system $\delta k+\beta$, where $\delta = 3, \beta = 1$.This paper establishes that such a system converges to β as long as the conditions are suitable. A full description of the Collatz system is given and formula derived for the reverse number, synchronized elements(those with similar Collatz Sequence) and closed forms that can help ease with the computation.

One of the most important discoveries of this paper is that the Collatz Space is bounded by 1 at the end and by an odd multiple of 3 on the other end. From this a proof of the elusive conjecture can then be derived easily

1.INTRODUCTION

Dynamical systems are sensitive to initial conditions therefore the Collatz Mechanism is one of the best to study such phenomena. In addition, it is consistent with the idea of attractors,[4].

The mathematical methods used in this paper are purely original works. However, there are many ways of mathematically describing the Collatz Problem such as logarithmic methods and stochastic stopping times done with statistical tools.More comprehensive material can be found in [5]. A comparison between the various methods may be done in a future paper.

Terrence Tao using logarithm methods proved that almost all orbits attain bounded values,[6].

This paper starts with a description of how the Collatz Mechanism works, derives a formula for the reverse number and goes on to give a closed form for the $n'th$ number in the sequence. In the last Chapter, I give a generalization of $\delta k+\beta$ systems and how they converge or diverge .

1.1 PROBLEM STATEMENT

The Collatz conjecture states that if you randomly pick a positive integer k. If k is even, divide it by 2, otherwise (odd), multiply it by 3 then add 1.If you repeat this procedure long enough, eventually the end result of k will always be 1. More information can be found in [1]

1.2 COLLATZ SEQUENCES

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Let's define a function f that satisfies the Conjecture so that by $f(k)$ we mean repeatedly applying the statement. For instance $f(5) = 16$ for one iteration. An Even Number k, is of the form $2r$ by definition, where $r \in 1, 2, 3..., n$. The consequence of this is that any even number can be reduced to an odd number. This helps avoid redundancies and reduces the scope of our analysis to $3k + 1$ for odd k. So $f(k)$ can be used to mean- getting the next odd number, γ in the Collatz Sequence $f(k) = 2c \times \gamma$

For instance

$3 \times 1 + 1 = 4 = 2^2 \times 1$

$3 \times 3 + 1 = 10 = 2^1 \times 5$

$3 \times 5 + 1 = 16 = 2^3 \times 1$

. . .

$3 \times k + 1 = \alpha + 6\beta = 2^c \times \gamma$

Where $\alpha = 4$, the first element of the arithmetic series $A = 4, 10, 16, 22...$
β, γ, k, c are elements of N , the set of counting numbers and k, γ is strictly odd.
Sample Collatz Sequence starting with 15 :

$\{f(15)\} = \{f(15), ..., 1\} = \{15,23,35,53,5,1\}$

1.3 IMPORTANT RESULTS.

From $3k + 1 = 2^c \times \gamma$ above,

$$\gamma = \tfrac{3k+1}{2^c}$$

$$= \tfrac{2k+k+1}{2^c} = \tfrac{2k}{2^c} + \tfrac{k+1}{2^c} = \tfrac{2k}{2^c} + \tfrac{(k-1)+2}{2^c}$$

$$= \tfrac{k}{2^{c-1}} + \tfrac{k-1}{2^c} + \tfrac{2}{2^c}$$

When $c = 1$ then,

$$\gamma = k + \tfrac{k-1}{2} + 1$$

Notice $c \neq 0$ at any instance.

One good finding from this is that the next number in the sequence depends on $\frac{k-1}{2}$. This will come handy in elaborating other concepts like synchronization and chaos.

Lemma 1.1

Two numbers γ_1 and γ_2 are synchronized if they have the same Collatz sequences .

This only happens when they satisfy any of the below conditions.

1. Synchronization by factorization

If $\gamma_1 = \gamma_2 \times 2^c$
So $\gamma_1 = 15$ and $\gamma_2 = 15 \times 2^c, c \in \mathbf{N}$ then γ_i are synchronized
In this case γ_1 and γ_2 don't have to be exclusively odd numbers.

$\{f(15)\} = \{23, 35, 53, 5, 1\}$

$\{f(30)\} = \{23, 35, 53, 5, 1\}$

$\{f(61440)\} = \{23, 35, 53, 1\}$

2. Synchronization by modularization.

For any two odd numbers γ_1 and γ_2 , they are synchronized if and only if $f(\gamma_1) = f(\gamma_2)$ with $\gamma_1 \neq \gamma_2$ .This happens when 2 numbers share a modulus by $f(\gamma_1)$

Mathematically defined by:-

$$\gamma_1 = 4\gamma_2 + 1$$

Observe the sequence generated by 7, 29, 117....

$\{f(7)\} = \{11, 17, 13, 5, 1\}$
$\{f(29)\} = \{11, 17, 13, 5, 1\}$
$\{f(117)\} = \{11, 17, 13, 5, 1\}$
$\{f(469)\} = \{11, 17, 13, 5, 1\}$

This is because:
29 modulus 11 = 7
117 modulus11 = 7
469 modulus 11 = 7

Lemma 1.2

Any Collatz sequence with odd constants γ_i i.e. $\{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n\}$ has a reverse function, $f^{\leftarrow}$ that gives the number preceding it but with a boundary condition.
.

Mathematically expressed as :

$$f^{\leftarrow}(\gamma_i) = \frac{4\gamma_i - c}{3} = \gamma_{i-1}$$

given that γ_i is not divisible by 3 and $c = 4\gamma_i mod3$

For example when $\gamma_i = 35$

$$f^{\leftarrow}(35) = \frac{4\times 35-2}{3} = 23 \times 2^1$$

23 precedes 35 in the Sequence 23, 35, 53,5, 1
Synchronization means an odd number has many revers as a solution because $4 \times 23 + 1 = 93$, which generates 93, 35,53,5,1
All successive $4\gamma_i + 1$ combinations of 23 are reverse elements of 35

$f^{\leftarrow}(35) = 23, 93, 373, 1493, ...$

Lets name 23 as the 'Collatz prime reverse number'.This number doesn't have to a prime number in the literal sense but all Synchronized sequences share such a number as their modulus.

93 mod 35 = 23
373 mod 35 = 23
1493 mod 35 = 23

Proposition 1.1

The Collatz Conjecture may be proved by finding the entire set of odd numbers in reverse beginning with the integer 1 only.

Step 1:

Obtain the reverse of 1.

$f^{\leftarrow}(1) = \frac{4\times 1-1}{3} = 1$

by lemma 1.2

$f^{\leftarrow}(1) = 1, 5, 21, 85, 341, ...$

Step 2

Arrange all the numbers next to 1 as follows:
1, 1
5 ,1
21,1
85,1
341,1

Step 3

All numbers that are not divisible by 3 have a reverse Collatz number that is consistent with the Collatz function.

So derive
$f^{\leftarrow}(5)$, $f^{\leftarrow}(85)$, $f^{\leftarrow}(341)$, ...
Which gives:

$f^{\leftarrow}(5)\} = 3, 13, 53, 213, 853, ...$
$f^{\leftarrow}(85)\} = 113, 453, 1813, 7253, ...$
$f^{\leftarrow}(341)\} = 227, 909, 3637, 14549, ...$

So we have:

1, 1
3, 5 ,1
13,5,1
21,1
53,5, 1
113, 85,1
453, 85,1
1813, 85,1
7253, 85,1
227, 341,1
909, 341,1
3637, 341,1
.. ..

Step 4

Repeat step 3 for all numbers not divisible by 3.

Doing this in the long run is similar to finding the Collatz sequence of all odd numbers divisible by 3.

3, 5, 1
9, 7, 11, 17, 13, 5, 1
15, 23, 35,53, 5,1
21, 1
27, 41,31, 47,71, ..., 61, 23,5, 1.
33, 25, 19, 29,11, 17, 13, 5, 1
39, 59, 89, 67, 101, 19, 11, 17, 13, 5, 1
3+6(k-1), ..., ..., ..., 1
3+6k .... , ... ,...., ...1

If either of these sets contain all the odd numbers, then the Collatz conjecture is true.

Corollary 1.1

The Collatz space of odd numbers is bounded by two numbers such that $(c_1, ..., c_n)$ where $c_1 = 3 + 6k, k \in N$ and $c_n = 1$

1.4 CLOSED FORM

Consider the following diverging sequence of even numbers.

28, 88, 272, 832, 2560, 8192

If its elements are factorized completely by 2 then we obtain the following sequence of odd numbers,
7, 11, 17, 13, 5, 1

The first sequence can be obtained as follows

Step 1

$3 \times 9 + 1 = 28$

Step 2

Find a number α of the form 2^c that factorizes 28 completely , which is 2^2
$2^2 \times 7 = 28$

Step 3

Solve $3 \times 28 + 2^2 = 88$

Step 4

Find another α that factorizes 88 completely , which is 2^3
Then solve $3 \times 88 + 2^3 = 272$

Step 5

Repeat the above procedure.

Doing this repeatedly shows a closed form must exist for any forward sequence.

Such a closed form can be generalized into:

$$\gamma_i = (((3\gamma_1 + \alpha_1)\delta + \alpha_2))3 + \alpha_3)3.. + \alpha_{n-1})3 + \alpha_n)$$

A closed form also exists form other for other $\delta k + \beta$ systems where $\delta \neq 3, \beta \neq 1$. Such will be discussed after we're done with the Collatz problem.

The sequence converges to 1 because:

$$\lim_{n\to\infty} \gamma_i = (((3\gamma_1 + \alpha_1)\delta + \alpha_2))3 + \alpha_3)3... + \alpha_{n-1})3 + \alpha_n)$$

$$= 2^r$$

1.5 CHAOS, DENSITY AND SYNCHRONIZATION

Previously it was established that $\gamma = \frac{k}{2^{c-1}} + \frac{k-1}{2^c} + \frac{2}{2^c}$ for one iteration and that the value of γ depends on $\frac{k-1}{2^c}$

Corollary 1.1

Synchronization is a special case of $\frac{k-1}{2^c}$ where $c = 2$
Proof:
Suppose we have two numbers such that:

$$k_1 = \frac{k_2-1}{2^2}$$

$$4k_1 = k_2 - 1$$

$$4k_1 + 1 = k_2$$

Hence k_1 and k_2 are synchronized by definition

15 and 61 are synchronized,
{f(15)} = { 23,35, 53,5,1}
{f(61)} = { 23,35, 53,5,1}

For non synchronized numbers we can observe how the sequences behave. So by 'k-1' decomposition it means to repeat the procedure until you arrive at 1. On the Left Hand Side is the k-1 decomposition and on the RHS is its Collatz sequence

a.
LHS RHS
23 = 35, 53, 5, 1
11 = 17, 13, 5, 1
5 = 1
1 = 1

b.
53 = 5,1
13 = 5,1
1 = 1

c.

247 =371,557,209,157,59,89,67,101,19,29,11,17,13,5,1,
123 =185,139,209,157,59,89,67,101,19,29,11,17,13,5,1,
61 = 23, 35, 53, 5, 1,
15 = 23, 35, 53, 5, 1,
7 = 11, 17, 23, 13,5, 1
3 = 5, 1
1 = 1

Synchronization occurs along the k-1 rows at some point . Various numbers have different densities for instance:

343 = 515, 773, 145, 109 , 41 ,..., 325, 61, 23, 35, 53, 5, 1
171 = 257, 193, 145, 109, 41, ..., 325, 61, 23, 35, 53, 5, 1
85 = 1
21 = 1
93, 145, 109, 41,..., 325, 61, 23, 35, 53, 5, 1
85 = 1
21 = 1
1 = 1

A dense Set

All members of $(2^i - 1)$,$i = 1, 2, 3, ..., n$ are successive k-1 decompositions
1, 3, 7, 15, 31, 63, 127, 255, 511
That means they do not synchronize right away after their first iteration.
Synchronization still occurs but it is not random as with other sets

63 = 95,143,215,323,485,91, ...,325,61,23,35,53,5,1
31 = 47,71,107,161,121,91,...,325,61,23,35,53,5,1
15 = 23,35,53,5,1
7 = 11,17,13,5,1
3 = 5,1
1 = 1

The size of the Collatz Sequence gets larger and larger as $(2^i - 1)$ increases. This can be a handy tool to show divergence.

CHAOTIC MATRIX

Suppose a matrix A can be constructed from the dense set in such a way that we can interpolate a leading diagonal with respect to the rows as shown below

$$\begin{bmatrix}
1 \\
3 & 5 & 1 \\
7 & 11 & 17 & 13 & 5 & 1 \\
15 & 23 & 35 & 53 & 5 & 1 \\
31 & 47 & 71 & 107 & 161 & 121 & 91 \ldots 53 & 5 & 1 \\
63 & 95 & 143 & 215 & 323 & 485 & 91 \ldots 35 & 53 & 5 & 1 \\
. & & & & & \ldots \\
. & & & & & & \ldots. \\
& & & & & & & \ldots. \\
\kappa & & & & & & & & \alpha_{n-1} \\
2\kappa+1 & & & \ldots. & & & & & & \boldsymbol{\alpha}_n
\end{bmatrix}$$

If the empty entries are filled with 1 then,

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
7 & 11 & 17 & 13 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
15 & 23 & 35 & 53 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
31 & 47 & 71 & 107 & 161 & 121 & 91 & 137 & 103 & 155 & \ldots 35 & 53 & 5 & 1 & 1 & 1 & 1 & 1 & 1 \\
63 & 95 & 143 & 215 & 323 & 485 & 91 & 137 & 103 & 155 & \ldots 35 & 53 & 5 & 1 & 1 & 1 & 1 & 1 & 1 \\
127 & 191 & 287 & 431 & 647 & 971 & 1457 & 1093 & 205 & 77 & 29 & 11 & 17 & 13 & 5 & 1 & 1 \\
255 & & & & & & \ldots \\
. \\
. & & & & & & & & \ldots. \\
. \\
\boldsymbol{k} & & & & & & & & & & \boldsymbol{\alpha}_{n-1} \\
2\boldsymbol{k}+1 & & & & & & & & & & & \alpha_n \\
\end{bmatrix}
$$

The diagonal can be trailed as follows

$$\begin{bmatrix} \alpha_0 & & & & & \\ & \alpha_1 & & & & \\ & & \alpha_2 & & & \\ & & & \cdots & & \\ & & & & \cdots & \\ & & & & & \alpha_n \end{bmatrix}$$

$\alpha_0 = 1$
$\alpha_1 = \alpha_0 \times 3 + 2$
$\alpha_2 = \alpha_1 \times 3 + 2$
$\alpha_n = \alpha_{n-1} \times 3 + 2$

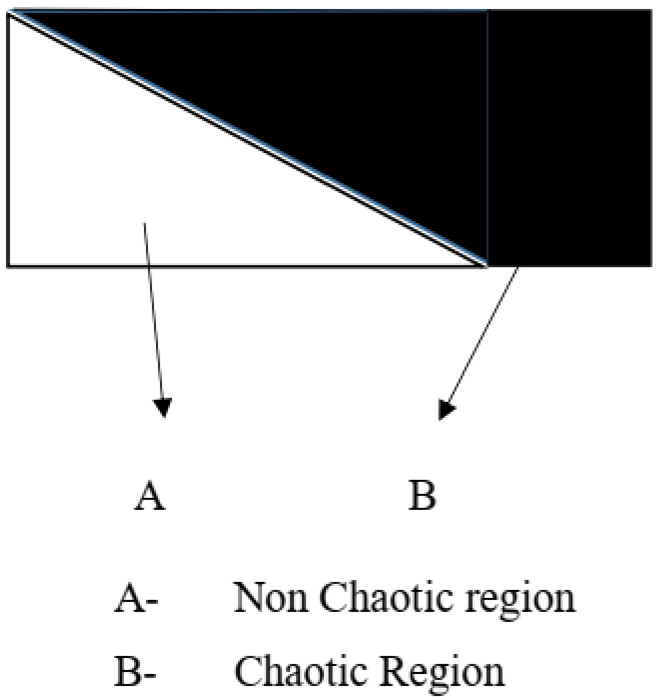
So The diagonals are 1, 5, 17, 53, 161, 485, 1457,...,
This series of diagonal entries is divergent but consistent with the above formula.

Corollary 1.2

The elements of the dense set synchronize if and only if the sequence has passed its diagonal entry. Thus, the chaos and randomness is distributed with respect to the diagonal.

Implication

It'd take a very very long time to disprove the Collatz Conjecture using brute force because we expect the elements between $2^n - 1$ and the corresponding α_n to be very many. In addition we'd have to prove the chaotic region also. A visual representation is shown below.



Corollary 1.3

Elements in the non-chaotic region maintain their k-1 decomposition. This means that $\frac{k-1}{2}$ is located directly above k and because of that we can reproduce a replica of region A without using the Collatz mechanism.

1.6 GENERALIZED CONVERGENCE OF COLLATZ-LIKE SYSTEMS

The discovery that $\delta k{+}1$ System for $\delta = 3$ 'absolutely' converges to $2^c, c \in \mathbf{N}$ for a given closed form of a Collatz System means that closed forms of systems of the form $\delta k + \beta$ for $\beta \in \mathbf{N}$ exist.

In this section a little more technicality will be used and instead of focusing on an odd γ_i we'll have a general case where the numbers k_i will be chosen indiscriminately as long as they maintain the underlying mechanism. Then we can create exponentially many combinations and thus an excellent dynamical system.

Let the dynamical system be denoted $\varphi(k)$ and defined by the conditional functions

$$\varphi(k) = \{^{\delta k+\beta, k\in\theta}_{g(k), k\in\mathbf{E}}$$

Where θ represents the set of Odd numbers and E the set of even numbers, hence $\theta \bigcup E = \mathbf{F}$, where $\mathbf{F}$ is the set $1, 2, 3, ...n$ with $\beta, \delta \in \mathbf{F}$

Let another function $g(k)$ denote the largest exponential of 2 that can completely factorize an even number k. So that the number is rewritten as $k = 2^n p$ then, $g(k) = 2^n$ for $n \in \mathbf{F}$, $p \in \theta$

The previous concepts applied to the Collatz Systems also occur in $\varphi(k)$ Systems like synchronization, the reverse numbers, density, convergence(divergence) and Closed Forms.

1.6.1 Convergence.

Lemma 1.3

Let $k_i = k_1, k_2, k_3, ..., k_n$ denote elements in a sequence that satisfies the conditional function with $\beta > 0$ then,

The Closed Form of $\varphi(k_i)$ is given by:

$$\varphi(k_i) = (((..(\delta k_1 + \beta g(k_1))\delta + \beta g(k_2))\delta + \beta g(k_3))\delta ... + \beta g(k_n))$$

With a Convergence of the form

$$\lim_{n\to\infty} \varphi(k_i) = (((..(\delta k_1 + \beta g(k_1))\delta + \beta g(k_2))\delta + \beta g(k_3))\delta ... + \beta g(k_n)) = \beta 2^m$$

In the Collatz problem $\beta = 1$ so our theorem holds because it converges to 2^m

1.6.2 Behaviour of the System $\beta \leq 0$

Lemma 1.3 is perhaps an overstatement when we consider integers of β less than or equal to zero .

The series maintains its closed form but the convergence rule breaks down for some integers. Generally,

$$\lim_{n\to\infty} \varphi(k_i) = (((..(\delta k_1 + \beta g(k_1))\delta + \beta g(k_2))\delta + \beta g(k_3))\delta ... + \beta g(k_n))$$

$$\neq \beta 2^m$$

in-fact the formula is given by :

$$\lim_{n\to\infty} \varphi(k_i) = ((..(\delta k_1 + \beta g(k_1))\delta + \beta g(k_2))\delta + \beta g(k_3))\delta ... + \beta g(k_n))$$

$$= (k_{n+1}g(k_{n+1})) \text{ with } \beta \neq k_{n+1}$$

Suitability of Convergence depends on the following:

1.The value of β
2. If δ allows for a consistent reverse system, thus guaranteeing forward and backward uniqueness, irregardless of how haphazard the system is.

1.6.3 Types of Convergence

1. Point Convergence

The sequence converges to a point if and only if

$$\lim_{n\to\infty} \varphi(k_i) = \{a_1, a_2, a_3, ..., a_n\}$$

where $\{a_1 = a_2 = a_3, ... = a_n\}$

The Collatz 3k+1 , k+1 all converge to to 1 because $\beta = 1$

2. Sequence Convergence

$\beta = 1$ doesn't imply convergence because 5k+1 doesn't converge to a point for some numbers.

$\beta = -1$ converges to 1 for $k - 1$

A series converges to a sequence if and only if

$$\lim_{n\to\infty} \varphi(k_i) = \{a_1, a_2, a_3, ..a_n\}$$

where, $\{a_1 \neq a_2, a_2 \neq a_3, a_3 \neq a_{4,}..., a_{n-1} \neq a_n\ \}$ but $a_1 = a_n$

Example with 3k -1 for k = 17

Seq = 17, 25, 37, 55, 41, 61, 91, 17, 25, 37, 55, 41, 61, 91,17...

k= 5
$\{5\} = \{5, 7, 5, 7, ..., 5\}$

Uniqueness and Convergence

For a sequence expanded fully $\{k_1, k_2, k_3...k_n, \eta_1, \eta_2, \eta_3, ..., \eta_n\}$ with the only repetition accepted for the first element to recur and then the series terminates.
If $k_1 = \eta_1$ then k_1 is a convergence point with unique $\eta_2, \eta_3, ..., \eta_{n-1}$.Otherwise, if $k_1 \neq \eta_1$ then all k_i are unique.

A good example from the previous sequence is $\{17, 25, 37, 55, 41, 61, 91, 17\}$. All elements but the first and the last are unique.

Any unique element in the sequence is also a convergence point for that sequence
The area of convergence is very crucial in understanding the sequence.

Consider the sequence generated by $k_1 = 33$ that eventually approaches $\eta_1 = 61$, 61 is an element of $\{17\}$ series, so it's a convergence point as stated earlier. Observe:
$\{33, 49, 73, 109, 163, 61, 91, 17, 25, 37, 55, 41, 61, \}$
Therefore any sequence that approaches in any of those elements converges to a sequence.

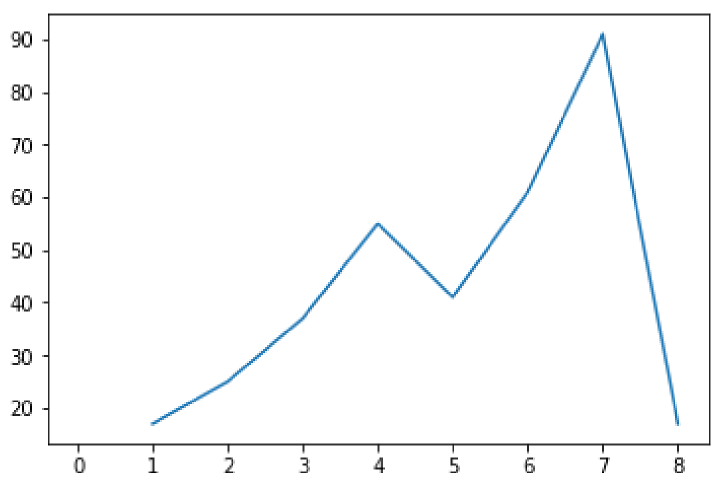
Applications

1. Higher Closed forms can make excellent cryptographic systems especially where there's a lot of constants and fairly long sequences that repeat themselves. With a little tweaking it could suffice, [2]. Hence chaos can be harnessed to produce encryption keys.

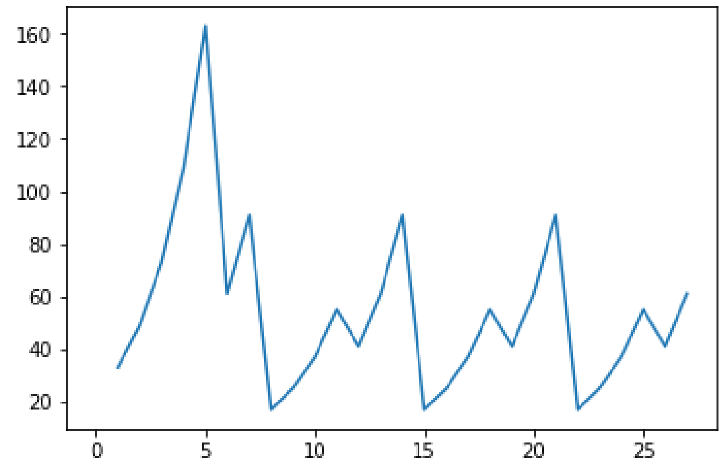
Consider a proof of concept using (5,7,5,...7)

10,112, 320,224, 5120, 3584,....
This sequence when factorized by 2 gives 5,7,5,...7

2. Another application can be used to study Stock prices and other derivatives that fluctuate sharply but their expected value after time t is the current value.[5]



3. A third example would be used to show decay that doesn't get to 0 over time but approaches an equilibria, [3,4]



4. The Collatz system itself can be used in epidemiology to study scenarios where we expected after , a time t the infection rate or the death rate might approach as small number as with the eradication of diseases like small pox once a vaccine is available.

CONCLUSION

In this paper, most of the phenomena behind the chaos in the Collatz System is defined mathematically. A reverse formula for all integers is obtained so as to give a closure on the Collatz Space. The Collatz Conjecture must have a proof because it is back and forward consistent. Finally, a brief analysis of $\delta k + \beta$ problems is discussed showing their divergence or convergence.

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