Criterion for the Riemann Hypothesis

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To my mother

Abstract. Let $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $M_x = \prod_{q \leq x} q$ be the product extending over all prime numbers q that are less than or equal to $x \geq 2$. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the Riemann hypothesis. We state that if the Riemann hypothesis is false, then there exist infinitely many natural numbers x such that the inequality $R(M_x) < \frac{e^{\gamma}}{\zeta(2)}$ holds, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. In this note, using our criterion, we prove that the Riemann hypothesis is true.

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1. Introduction

The Riemann hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm.

Proposition 1.1. *For every* $x > 1$ [9, Theorem 4 (3.15) pp. 70]*:*

$$\theta(x) < \left(1 + \frac{1}{2 \cdot \log x}\right) \cdot x.$$

The following property is based on natural logarithms:

Proposition 1.2. *For* $x > -1$ [6, pp. 1]*:*

$$\log(1 + x) \leq x.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.3. *We define* [1, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k *is the* k*th prime number. By definition, we have*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n *denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ *is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.*

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\begin{aligned} \gamma &= \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k}\right) \\ &= \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx. \end{aligned}$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Franz Mertens discovered some important results about the constants B and H (1874) [7]. The number $B \approx 0.26149$ is the Meissel-Mertens constant where $\gamma = B + H$ [7].

Proposition 1.4. *We have* [3, Lemma 2.1 (1) pp. 359]*:*

$$\sum_{k=1}^{\infty} \left(\log\left(\frac{q_k}{q_k - 1}\right) - \frac{1}{q_k}\right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows [8, pp. 379]:

$$u(x) = \sum_{q > x} \left(\log\left(\frac{q}{q - 1}\right) - \frac{1}{q}\right).$$

On the sum of the reciprocals of all prime numbers not exceeding x, we have:

Proposition 1.5. *For $x > 1$ [9, Theorem 5 (3.17) pp. 70]:*

$$-\frac{1}{2 \cdot \log^2 x} < \sum_{q \leq x} \frac{1}{q} - B - \log \log x.$$

Definition 1.6. We define a function about the partial sum of the reciprocals of primes:

$$T(x) = \sum_{q \leq x} \frac{1}{q} - B - \log \log \theta(x).$$

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function where $q \mid n$ means the prime q divides n. For $x \geq 2$, a natural number M_x is defined as

$$M_x = \prod_{q \leq x} q.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. We say that $\mathsf{Dedekind}(x)$ holds provided that

$$R(M_x) \geq \frac{e^{\gamma}}{\zeta(2)}.$$

Definition 1.7. We define a new constant:

$$J = \frac{(\log(\zeta(2)) - H)}{(\gamma - \log(\zeta(2)))}.$$

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [4]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [5]:

$$f(x) = \Omega_R(g(x)) \ \textit{as } x \to \infty \ \textit{if} \ \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$
$$f(x) = \Omega_L(g(x)) \ \textit{as } x \to \infty \ \textit{if} \ \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_{\pm}(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as [11]:

$$f(x) = \Omega_+(g(x)) \ \textit{if} \ \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis could be false. The following is a key Lemma.

Lemma 2.1. *If the Riemann hypothesis is false, then there exist infinitely many natural numbers x for which* $\mathsf{Dedekind}(x)$ *fails (i.e.* $\mathsf{Dedekind}(x)$ *does not hold).*

Proof. The function g is defined as [10, Theorem 4.2 pp. 5]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

The Riemann hypothesis is false whenever there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ [10, Theorem 4.2 pp. 5]. It was proven the following bound [10, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}.$$

For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [8, Theorem 3 pp. 376], [2, (5.5) pp. 111]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

If the Riemann hypothesis is false then there exists a real number b with $0 < b < \frac{1}{2}$ such that, as $x \to \infty$ [8, Theorem 3 (c) pp. 376], [2, Theorem 5.29 pp. 131],

$$\log f(x) = \Omega_{\pm}(x^{-b}).$$

Actually Nicolas proved that $\log f(x) = \Omega_{\pm}(x^{-b})$, but we only need to use the notation Ω_{+} in this proof under the domain of natural numbers. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon \ \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is also $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible values of $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon \ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Note that, the variable k disappears in our previous expression because of we do not need it anymore. In this way, if the Riemann hypothesis is false, then there exist infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{1}{\sqrt{x_0}} > \frac{2}{x_0}$ for $x_0 \geq 5$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. □

3. Essential Sums

These are essential sums.

Lemma 3.1.

$$\sum_{q \leq x} \left(\frac{1}{q} - \log \left(1 + \frac{1}{q} \right) \right) = \log \left(\prod_{q \leq x} \frac{q^2}{q^2 - 1} \right) - H + u(x).$$

Proof. We obtain that

$$\begin{aligned}
&\log \left(\prod_{q \leq x} \frac{q^2}{q^2 - 1} \right) - H + u(x) \\
&= \sum_{q \leq x} \log \left(\frac{q^2}{(q^2 - 1)} \right) - H + u(x) \\
&= \sum_{q \leq x} \left(\log \left(\frac{q^2}{(q - 1) \cdot (q + 1)} \right) \right) - H + u(x) \\
&= \sum_{q \leq x} \left(\log \left(\frac{q}{q - 1} \right) + \log \left(\frac{q}{q + 1} \right) \right) - H + u(x) \\
&= \sum_{q \leq x} \left(\log \left(\frac{q}{q - 1} \right) - \log \left(\frac{q + 1}{q} \right) \right) - H + u(x) \\
&= \sum_{q \leq x} \left(\log \left(\frac{q}{q - 1} \right) - \log \left(1 + \frac{1}{q} \right) \right) - \sum_{q \leq x} \left(\log \left(\frac{q}{q - 1} \right) - \frac{1}{q} \right) \\
&= \sum_{q \leq x} \left(\log \left(\frac{q}{q - 1} \right) - \log \left(1 + \frac{1}{q} \right) - \log \left(\frac{q}{q - 1} \right) + \frac{1}{q} \right) \\
&= \sum_{q \leq x} \left(\frac{1}{q} - \log \left(1 + \frac{1}{q} \right) \right)
\end{aligned}$$

by Propositions 1.3 and 1.4. □

Lemma 3.2.

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log \left(1 + \frac{1}{q_k} \right) \right) = \log(\zeta(2)) - H.$$

Proof. This is a consequence of applying the Lemma 3.1 when x tends to infinity by Propositions 1.3 and 1.4. □

Lemma 3.3.

$$\sum_{q > x} \left(\frac{1}{q} - \log \left(1 + \frac{1}{q} \right) \right) = \log \left(\prod_{q > x} \frac{q^2}{q^2 - 1} \right) - u(x).$$

Proof. By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\sum_{q>x}\left(\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right) &= \sum_{k=1}^{\infty}\left(\frac{1}{q_k}-\log\left(1+\frac{1}{q_k}\right)\right)-\sum_{q\leq x}\left(\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right)\\ &= \log(\zeta(2))-H-\log\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)+H-u(x)\\ &= \log\left(\prod_{q>x}\frac{q^2}{q^2-1}\right)-u(x).\end{aligned}$$

□

4. Main Insight

This is the main insight.

Lemma 4.1. *The inequality* $\frac{\prod_{q\leq x} e^{\frac{1}{q}}}{\log\theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^{1+J}$ *holds for large enough* $x \in \mathbb{N}$.

Proof. By Proposition 1.4, the inequality

$$\frac{\prod_{q\leq x} e^{\frac{1}{q}}}{\log\theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^{1+J}$$

is the same as

$$\sum_{q\leq x}\left(\frac{1}{q}\right)-B-\log\log\theta(x) \geq H+J\cdot\gamma-(1+J)\cdot\log(\zeta(2))$$

after of applying the logarithm to the both sides and distributing the terms. In addition, we can see that

$$\begin{aligned}\log\log\theta(x) &< \log\log\left(\left(1+\frac{1}{2\cdot\log x}\right)\cdot x\right)\\ &= \log\left(\log\left(1+\frac{1}{2\cdot\log x}\right)+\log x\right)\\ &= \log\left((\log x)\cdot\left(1+\frac{\log\left(1+\frac{1}{2\cdot\log x}\right)}{\log x}\right)\right)\\ &= \log\log x+\log\left(1+\frac{\log\left(1+\frac{1}{2\cdot\log x}\right)}{\log x}\right)\\ &\leq \log\log x+\frac{\log\left(1+\frac{1}{2\cdot\log x}\right)}{\log x}\\ &\leq \log\log x+\frac{1}{2\cdot\log^2 x}\end{aligned}$$

by Propositions 1.1 and 1.2. So, we would have

$$\sum_{q\leq x}\left(\frac{1}{q}\right)-B-\log\log x-\frac{1}{2\cdot\log^2 x}\geq H+J\cdot\gamma-(1+J)\cdot\log(\zeta(2)).$$

That is,

$$\sum_{q\leq x}\left(\frac{1}{q}\right)-B-\log\log x-\frac{1}{2\cdot\log^2 x}-u(x)\geq H-u(x)+J\cdot\gamma-(1+J)\cdot\log(\zeta(2)).$$

after subtracting $u(x)$ to the both sides of the inequality. By Proposition 1.5, we can check that

$$-\frac{1}{2\cdot\log^2 x}-\frac{1}{2\cdot\log^2 x}-u(x)\geq H-u(x)+J\cdot\gamma-(1+J)\cdot\log(\zeta(2)).$$

By Lemma 3.1, we obtain that

$$\sum_{q\leq x}\left(\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right)-\frac{1}{\log^2 x}-u(x)\geq J\cdot(\gamma-\log(\zeta(2)))-\log\left(\prod_{q>x}\frac{q^2}{q^2-1}\right).$$

By Definition 1.7 and Lemma 3.2, we deduce that the previous inequality holds for large enough $x\in\mathbb{N}$ due to

$$\lim_{x\to\infty}\left(\sum_{q\leq x}\left(\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right)-\frac{1}{\log^2 x}-u(x)\right)=\log(\zeta(2))-H. \qquad \square$$

5. Primary Lemma

The following is a primary Lemma.

Lemma 5.1. *If $T(x)\geq 0$ holds for $x\in\mathbb{N}$ then* $\mathsf{Dedekind}(x)$ *also holds.*

Proof. If $\mathsf{Dedekind}(x)$ holds for $x\in\mathbb{N}$, then

$$R(M_x)\geq\frac{e^\gamma}{\zeta(2)}.$$

That is the same as

$$e^{\log(\zeta(2))-H}\cdot\prod_{q\leq x}\left(1+\frac{1}{q}\right)\geq e^B\cdot\log\theta(x)$$

by Proposition 1.4. By Lemmas 3.1, 3.2 and 3.3, we deduce that

$$e^{\log\left(\prod_{q>x}\frac{q^2}{q^2-1}\right)-u(x)}\cdot\prod_{q\leq x}e^{\frac{1}{q}}\geq e^B\cdot\log\theta(x).$$

By Definition 1.6, that is equivalent to

$$T(x)\geq\left(u(x)-\log\left(\prod_{q>x}\frac{q^2}{q^2-1}\right)\right).$$

after applying the logarithm to the both sides and distributing the terms. By Proposition 1.2 and Lemma 3.3, we deduce that

$$u(x) - \log\left(\prod_{q>x} \frac{q^2}{q^2-1}\right) \leq 0.$$

To sum up, we can assure that $\mathsf{Dedekind}(x)$ holds whenever $T(x) \geq 0$ holds for $x \in \mathbb{N}$.

□

6. Main Theorem

This is the main theorem.

Theorem 6.1. $\mathsf{Dedekind}(x)$ *always holds for large enough* $x \in \mathbb{N}$.

Proof. By Lemma 4.1, the inequality

$$\frac{\prod_{q\leq x} e^{\frac{1}{q}}}{\log\theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^{1+J}$$

holds for large enough $x \in \mathbb{N}$. That would be

$$\frac{\prod_{q\leq x} e^{\frac{1}{q}}}{e^{B}\cdot\log\theta(x)} = e^{T(x)} \geq \frac{1}{e^{B}}\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right)^{1+J}$$

Hence, it is enough to show that

$$\frac{1}{e^{B}}\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right)^{1+J} \geq 1$$

holds to confirm that $\mathsf{Dedekind}(x)$ also holds by Lemma 5.1. By Definition 1.7, we would have

$$\frac{1}{e^{B}}\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right)^{J}\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right) \geq 1$$

which is

$$\frac{1}{e^{B}}\cdot e^{\log(\zeta(2))-H}\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right) \geq 1$$

since $x^{\frac{1}{\log x}} = e$ for $x > 0$. By Proposition 1.4, that would be

$$e^{\log(\zeta(2))-\gamma}\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right) = \left(\frac{\zeta(2)}{e^{\gamma}}\right)\cdot\left(\frac{e^{\gamma}}{\zeta(2)}\right) = 1 \geq 1$$

and therefore, the proof is done. □

7. Main Result

This is the main result.

Corollary 7.1. *The Riemann hypothesis is true.*

Proof. By Lemma 2.1, if the Riemann hypothesis is false, then there exists an infinite sequence of natural numbers x_i such that $\mathsf{Dedekind}(x_i)$ fails. This contradicts the fact that $\mathsf{Dedekind}(x)$ always holds for large enough $x \in \mathbb{N}$ according to the Theorem 6.1. By Reductio ad absurdum, the Riemann hypothesis must be true as a direct consequence of Lemma 2.1 and Theorem 6.1. □

8. Conclusions

Practical uses of the Riemann hypothesis include many propositions that are considered to be true under the assumption of the Riemann hypothesis and some of them that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. A proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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