Efficient Algorithm for Graph Isomorphism Problem

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ABSTRACT

In this research paper, polynomial time algorithms for graph isomorphism problem (i.e. effectively deciding whether two graphs are isomorphic) are discussed under some conditions. It is reasoned that the cardinality of graphs for which the conditions hold are much larger than those for which the conditions donot hold. Thus, a probabilistic polynomial time algorithm is designed. The algorithms are essentially based on linear algebraic concepts related to graphs. Also, some new results in spectral graph theory are discussed.

1. INTRODUCTION:

Directed/undirected, weighted/unweighted graphs naturally arise in various applications. Such graphs are associated with matrices such as weight matrix, incidence matrix, adjacency matrix, Laplacian etc. Such matrices implicitly specify the number of vertices/ edges, adjacency information of vertices (with edge connectivity) and other related information (such as edge weights). In recent years, there is explosive interest in capturing networks arising in applications such as social networks, transportation networks, bio-informatics related networks (e.g. gene regulatory networks) using suitable graphs. Thus, NETWORK SCIENCE led to important problems such as community extraction, frequent sub-graph mining etc. In many applications the problem of deciding whether two given graphs are isomorphic (i.e. the two graphs are essentially same upto relabeling the vertices) naturally arises. This research paper provides one possible solution to such a problem.

This research paper is organized in the following manner. In section 2, relevant research literature is briefly reviewed. In section 3, two polynomial time algorithms, to test if two graphs are isomorphic are discussed. In section 4, interesting results related to spectral graph theory are discussed. The research paper concludes in section 5.

2. REVIEW OF RESEARCH LITERATURE:

L. Babai recently claimed quasi-polynomial time algorithm for determining if two graphs are isomorphic [1] . This is the most recent contribution to the graph isomorphism problem. Specifically Babai showed that graph isomorphism problem can be solved in ($\exp((\log n)^{O(1)})$ time [2]. For the problem, the previous known best bound was $\exp\left(o(square\ root\ of\ (n \log n\ )\ )\right)$, where $'n'$ is the number of vertices (Luks, 1982, [9]).There are other research efforts which provide approximate solutions to the problem (i.e. approximate algorithms were designed) [3], [4],

[5],[6],[9]. Also, the problem of solving Graph Isomorphism has been attempted using the quadratic non-negative matrix factorization problem[15].

3. POLYNOMIAL TIME ALGORITHMS FOR GRAPH ISOMORPHISM PROBLEM (UNDER SOME CONDITIONS) :

We now briefly review relevant results from spectral graph theory.

3.1 **Spectral Graph Theory**: Spectral graph theory deals with the study of properties of a graph in relationship to the characteristic polynomial, eigenvalues and eigenvectors of matrices associated with the graph, such as its adjacency matrix or Laplacian matrix.

- An undirected graph has a symmetric adjacency matrix A and hence all its eigenvalues are real. Furthermore, the eigenvectors are orthonormal.

We have the following definition

Definition: An undirected graph's SPECTRUM is the multiset of real eigenvalues of its adjacency matrix, A. Graphs whose spectrum is same are called co-spectral.

Remark 1. It is well known that isomorphic graphs are co-spectral. But co-spectral graphs need not be isomorphic. Thus spectrum being same is only a necessary condition for graphs to be isomorphic (but not sufficient). Thus, it is clear that the eigenvectors of adjacency matrices of isomorphic graphs must be constrained in a suitable manner (orthonormal basis vectors of the symmetric adjacency matrices are some how related for isomorphic graphs).

3.2. Polynomial Time Algorithm to determine cospectral Graphs:

Lemma 1: The problem of determining if two graphs are Co-Spectral is in P (i.e. a polynomial time algorithm exists)

Proof: Since the elements of adjacency matrix are '0's and '1's, the characteristic polynomial of it is a polynomial with integer coefficients. Thus, there exists a polynomial time algorithm [7] (LLL algorithm) to compute the zeroes of such polynomial i.e. spectrum of associated graph. Thus the problem of determining if two graphs are cospectral is in P (class of polynomial time algorithms)……………………………………………………..Q.E.D.

Note: By Perron-Frobenius theorem, the spectral radius of an irreducible adjacency matrix (non-negative matrix) is real, positive and simple. Thus, to check for the necessary condition on isomorphic graphs, a first step is to determine if the spectral radius of two gaphs are exactly same.

Definition: Two graphs are isomorphic, if the vertices of one graph are obtained by relabeling the vertices of another graph.

3.3. Necessary and Sufficient Conditions: Isomorphism of Graphs:

3.3.1 Necessary Conditions: Isomorphism of Graphs.

• The following necessary conditions for isomorphism of graphs with adjacency matrices A, B can be checked before applying the following algorithm

• Check if Trace(A) = Trace(B) and if Determinant(A) = Determinant(B)

• Check if Spectral radius of A, B are same. This can be done using the Jacob's algorithm for computing the largest zero of a polynomial. Since the coefficients of characteristic polynomial are integers, we expect the computational complexity of this task to be smaller. If this step fails, all other zeroes need not be computed.

We now consider two separate Cases (based on the eigenvalues of the graphs i.e. the multiset of zeroes of characteristics polynomial of the associated adjacency matrices):

CASE I: Eigenvalues of Adjacency Matrix are all DISTINCT: It should be noted that there are several interesting graphs all of whose eigenvalues are distinct. In fact, given a fixed number of vertices, the number of graphs with distinct eigenvalues are more than the graphs with repeated eigenvalues.

To provide the necessary and sufficient conditions for graph isomorphism, we need the following well know Lemma from Linear Algebra.

Lemma 2: Every symmetric matrix has UNIQUE spectral representation (Eigen Decomposition) when all its eigenvalues are distinct.

Proof: Refer Linear Algebra book [11] Q.E.D.

Let the adjacency matrices of two given co-spectral graphs be A, B. Suppose the graphs are isomorphic i.e. there exists a permutation matrix P such that

$$B = P\ A\ P^T \text{......................}(1)$$

But, since A is a symmetric matrix, we have that

$$A = F\ D\ F^T$$

where D is the Diagonal matrix of eigenvalues of A and F is an UNIQUE *orthogonal matrix* (i.e. $F^T = F^{-1}$) *whose* columns are right eigenvectors of A. Also, we have that

$$B = \ G\ D\ G^T \text{.................}(2)$$

Thus, we must have that

$$G = P\ F \quad or \quad G\ F^T = P.$$

But, we know that a Permutation matrix, P must be DOUBLY STOCHASTIC and *there is precisely one* $p_{ij} = 1$ *in each row and each column.*

Thus, the orthogonal matrices G, F must be related by the above equation i.e. we need to check if P is doubly stochastic and if precisely *one* $p_{ij} = 1$ in each row and each column.
Therefore, the above condition is necessary and sufficient, when the eigenvalues are distinct real numbers

Hence, we have proved the following Lemma.

Lemma 3: Given the above Spectral Representations of adjacency matrices A, B (with unique G, F) of two graphs, they are isomorphic if and only if $G\,F^T = P$, where P is a Permutation matrix.

The computational complexity of checking the above condition leads to a polynomial time algorithm under some conditions. Based on the lemma 3, in the following sub-section, we summarize the steps of the algorithm to determine if two graphs are isomorphic, when the eigenvalues are distinct.

3.3.2 Polynomial Time Algorithm under some conditions: Algorithmic Steps: Proof of Correctness of the Algorithm: Computational Complexity:

(0) Det (λ I − A) is a polynomial with integer coefficients. Also, characteristic polynomial of B has integer coefficients. If the eigenvalues of A, B are not same, then they are not even cospectral and the algorithm stops. If they are cospectral, proceed to the following step to determine if they are isomorphic graphs.

Computational Complexity: A polynomial time algorithm (LLL Algorithm) [7] exists for this problem.

(1) Compute Spectral representation of adjacency matrices A, B of two given graphs:

$$A = F\,D\,F^T \;\; and \;\; B = \; G\,D\,G^T$$

where D is the common set of eigenvalues and F, G are the unique orthogonal matrices.

Computational Complexity: Efficient polynomial time algorithms exist for this problem when the eigenvalues are rational numbers. Effectively, eigenvectors of A, B are computed in polynomial time. Gaussian Elimination to compute every eigenvector requires atmost $O(N^3)$ Computation (additions, multiplications). Thus, in the worst case, this step requires $O(N^4)$ computations (additions, multiplications) for computing all the eigenvectors.

Note: Research effort motivated by Strassen's algorithm currently requires $O(N^{2.4})$

Computations. Thus the Step (2) requires smaller number of computations.

(2) Determine if $G\,F^T = P$, where P is a Permutation matrix i.e. if P is doubly stochastic and there is precisely *one* $p_{ij} = 1$ *in each row and each column.*

Computational Complexity: Efficient polynomial time algorithms are well known for this problem. This step requires $O(N^2)$ comparisions.

Note: If the eigenvalues of A, B are rational numbers, polynomial time algorithm definitely exists. Even if the eigenvalues are irrational numbers, it is possible that a polynomial time algorithm can be found.

Note: For step 1, Prof. Lovasz informed the author that if we assume that exact real arithmetic can be carried out, polynomial time algorithm exists. He also informed that if we can model the problem in an approximate computing model, polynomial time algorithm exists [8].

- **Graphs with Distinct Spectra** (i.e. eigenvalues of adjacency matrix are all distinct real numbers):

Lemma 4: For a given number of vertices, the number of graphs all of whose eigenvalues are distinct are larger than the number of graphs for which some of the eigenvalues are repeated.

Proof: In view of existence of polynomial time algorithm for determining graph isomorphism when the eigenvalues (of adjacency matrix) are distinct, we would like to determine the ratio of polynomials (of a fixed degree, N) with integer coefficients whose zeroes are distinct to those with repeated zeroes. This ratio can be readily determined from the following ratio i.e.

$$\frac{Number\ of\ polynomials\ of\ degree\ N\ with\ repeated\ zeroes}{Total\ Number\ of\ Polynomials\ of\ Degree\ N}.$$

(since the denominator is a sum of numerator and polynomials with distinct roots).
It is clear that the characteristic polynomial of N x N adjacency matrix (with { 0, 1 } elements) i.e. Det (λ I – A) = f (λ) is a polynomial, all of whose integer coefficients are bounded in absolute value by a constant 'n'.
We first determine the ratio for quadratic polynomials i.e. $N = 2$. It readily follows that since all the coefficients are bounded, they can assume $2\,n + 1$ values. Hence the above ratio becomes

$$\frac{2\,(2\,n+1\,)^2}{(2\,n+1\,)^3} = \frac{Number\ of\ Quadratics\ with\ repeated\ zeroes}{Total\ number\ of\ quadratics}.$$

The numerator is determined from the fact that the quadratic has repeated roots if and only if the discriminant is zero. More explicitly, suppose $a\,x^2 + b\,x + c$ is the quadratic polynomial all of whose coefficients are bounded in magnitude by n. The discriminant is zero if and only if $b^2 = 4\,a\,c$. For each possible pair of values for integers $\{\,a, c\,\}$, there are atmost 2 integer values for b. Consequently, there are atmost $2\,(2\,n+1\,)^2$ quadratics whose zeroes are repeated.

Now, we generalize the above argument for polynomial of an arbitrary degree N. In this case, the discriminant will form an algebraic surface in (N+1)-dimensional space. Hence, by an exactly analogus argument for N =2, there are atmost $(c)(n^N)$ polynomials with repeated roots. Also, there are a total of $(2\,n+1\,)^{N+1}$ polynomials of degree N, all of whose coefficients are bounded in magnitude by n. Thus, the ratio becomes

$$\frac{c\,n^N}{(2\,n+1\,)^{N+1}}.$$

From this ratio, it readily follows that, the derived ratio

$$\frac{Number\ of\ polynomials\ of\ degree\ N\ with\ repeated\ zeroes}{Number\ of\ Polynomials\ of\ Degree\ N\ with\ distinct\ roots}$$

will be strictly less than one.

It is clear that given the degree of polynomial N, the bound $'n'$ depends on N.

Thus, the above two ratios are

$$\theta(N,n\,) = \frac{Number\ of\ polynomials\ of\ degree\ N\ with\ repeated\ zeroes}{Total\ Number\ of\ Polynomials\ of\ Degree\ N}\quad and$$

$$\beta(N,n)\ = \frac{Number\ of\ polynomials\ of\ degree\ N\ with\ repeated\ zeroes}{Number\ of\ Polynomials\ of\ Degree\ N\ with\ distinct\ roots}.$$

It readily follows that

$$\theta(N,n\,) = \frac{1}{1+\beta(N,n)}.$$

The following corollary readily follows $Q.E.D.$

Corollary: $$\lim_{\substack{N\to\infty\\ n\to\infty}} \theta(N,n\,) = 0 \quad and \quad \lim_{\substack{N\to\infty\\ n\to\infty}} \beta(N,n\,) = \infty$$

Proof: Using Binomial Theorem, we have that

$$\theta(N,n\,) = \frac{C\ n^N}{(2n+1)^{N+1}} = \frac{c}{(\,2n+1\,)}\ \frac{n^N}{\sum_{j=0}^{N}\binom{N}{j}(2\,n\,)^j}.$$

Using the fact that limit of product of two real sequences is the product of limits of them, we have that

$$\lim_{\substack{N\to\infty\\ n\to\infty}} \theta(N,n\,) = 0 \quad and\ hence \quad \lim_{\substack{N\to\infty\\ n\to\infty}} \beta(N,n\,) = \infty. \qquad Q.E.D.$$

Note: More informally, when some of the zeroes are repeated, the discriminant of the characteristic polynomial is zero. Thus, since the discriminant is not constrained in the case of distinct eigenvalues, we expect the number of graphs with distinct eigenvalues to be larger than those graphs whose spectra contain repeated eigenvalues.

Note: Also, informally, the ratio of number of polynomials with repeated zeros divided by the number of polynomials with distinct zeros is less than one. This is because if we look at at the possible coefficients of an N^{th} degree polynomial that are bounded in absolute value by B. Then we are looking at lattice points in an $N+1$ dimensional box of side 2B+1. The polynomials with multiple zeros must have discriminant equal to zero. This means they must lie on some algebraic surface within the box and consequently their number must be of a lower order of magnitude.

Note: In view of the above discussion, given the number of vertices, say M, the number of graphs with distinct spectra (i.e. the adjacency matrix has distinct eigenvalues) are MUCH MORE than those graphs whose spectra is not distinct.

Note: In view of the above discussion, given two arbitrary graphs, a Probabilistic Polynomial Time algorithm (i.e. PP class) is designed to test if the given two graphs are isomorphic.

CASE II: Eigenvalues of A, B are NOT ALL DISTINCT (some eigenvalues are repeated):

We now consider the more general problem when the eigenvalues of A, B (that are all equal) are NOT DISTINCT.

- **Quadratic Non-Negative Matrix Factorization**: As discussed earlier, from equation (1), the problem boils to determine if a Permutation matrix P exists such that

$$B = P\,A\,P^T.$$

Such a problem is already being attempted using the approach based on Quadratic Non-Negative Matrix Factorization [15]. The results proposed for such a problem readily apply for determining isomorphism of two graphs.

- **Structured Quadratic Programming Problems**: There is another interesting way of looking at the equation (1). Let the unknown matrix P be given by (in terms of columns)

$$P = [\,P_1 \;\; P_2 \;\; \dots P_N\,]\,.$$

Since A, B are { 0, 1 } matrices, we have homogeneous, second degree equations (quadratic forms) in the elements of unknown matrix P with coefficients being { 0, 1 } and the bi-variate homogeneous polynomials being equated with values { 0, 1}. Further the variables (i.e. elements of P) are constrained to be { 0, 1 }. Hence, we have structured set of simultaneous quadratic programming problems. The problem boils down to testing if the { 0, 1 } solutions (if they exist) lead to a permutation matrix, P.

- **Algebraic Riccati Equation: Symmetric Permutation Matrix P**

  The quadratic matrix equation (non-linear) has resemblance to the Symmetric Algebraic Riccati Equation of the following form

$$X\,C\,X - A\,X - X\,A^T + B = 0$$

  (with compatible matrices X, C, A, B), where *the matrice B and C are symmetric and X is the unknown matrix*. As can be readily seen the matrix equation (1) is a *structured symmetric Algebraic Riccati equation* with P *being a symmetric unknown matrix and* $A \equiv 0$.

The known algorithms for solving such a Riccati equation may readily apply for testing isomorphism of two graphs for which P is a symmetric permutation matrix. Specifically, there are efforts to determine the non-negative matrix solutions of Riccati equation [16], [17]. It should be kept in mind that the solution of algebraic Riccati equation that is of interest to us is a structured { 0, 1 } matrix.

- **Explicit Solution when the Adjacency Matrices of the graphs are non-singular and are related through "Symmetric" Permutation Matrix: Algorithm 2: (If graphs are isomorphic, the algorithm declares them correctly).**

Lemma 5: Under the above assumptions, two graphs with adjacency matrices { B, C } (whose eigenvalues need NOT be distinct) are isomorphic if

$$X = [\, Matrix\ Square\ Root\ (\ B\ C\ )\,]\ C^{-1}$$

is a Permutation matrix.

Proof: We are interested in the solution of following MATRIX EQUATION:

$$X\ C\ X = B,\ where\ \{ C, B \}\ are\ adjacency\ matrices\ of\ graphs\ .$$

$$Multiplying\ on\ both\ sides\ of\ the\ equation\ by\ C,\quad we\ have\ that$$

$$(X\ C)(\ X\ C\ ) = \ B\ C\ .\ Hence,\quad it\ readily\ folllows\ that$$

$$X\ C = Matrix\ \ Square\ \ Root\ (\ B\ C\ ).$$

$$Now,\quad if\ \ C\ \ is\ \ non-singular, X\ \ can\ \ readily\ be\ \ computed\ \ as$$

$$X = [\, Matrix\ Square\ Root\ (\ B\ C\ )\,]\ C^{-1}$$

The matrix square root is unique only when BC is a positive definite matrix. In the case where the two graphs are isomorphic (with a symmetric permutation matrix P i.e. $P = P^T$ *and* $B = P\ C\ P$), *it readily follows that B C is a positive definite matrix (with B and C being non-singular matrices with the same set of eigenvalues*). It is possible that, the graphs are not isomorphic, but BC is a positive definite matrix. Q.E.D.

Remark 2: In view of the above three equivalent problems, the results available for solution of one problem can be utilized in the solution of other problems. For instance, when eigenvalues of A, B are equal and distinct, the algorithm discussed for graph isomorphism can be utilized in other problems.

- **Spectra of Graphs:**
  Babai showed that in a well-defined sense, Johnson graphs are the only obstructions to effective canonical partitioning [2]. In view of such fact, we focused on the spectra of Johnson graphs.

  (1) We determined that the eigenvalues of (adjacency matrix) (3,1) Johnson graph are { -1, -1, 2 }. Hence, some of the eigenvalues are repeated. Similarly, the eigenvalues of (4, 2) Johnson graph are { -2,-2,0,0,0, 4 }. Thus, once again, some of the eigenvalues are repeated. In view of such empirical evidence, we are led to the following conjecture.

  CONJECTURE: One or more eigenvalues of every Johnson graph are always repeated.

  We now provide an approach to prove the above conjecture:

- The spectrum of a Johnson graph is known to be given by the Eberlein polynomial. Hence, to prove the above conjecture, it is sufficient to prove that the DISCRIMINANT of the associated Eberlein polynomial is always zero.

- We now determine the spectra of fully connected graphs (cliques) without self loops at all the vertices. The adjacency matrix (with N vertices) is given by

  $$A = \bar{e}\bar{e}^T - I, \quad \text{where } \bar{e} \text{ is a column vector of ONES}.$$

  Hence, we have that, if $'\alpha'$ is an eigenvalue of A and $'\mu'$ is an eigenvalue of rank-one matrix, $\bar{e}\bar{e}^T$,

  $$\alpha = \mu - 1.$$

  Thus $\left(\textit{since the eigenvalues of } \bar{e}\bar{e}^T \textit{ are } \{N, 0,0, \dots ,0\} \right)$, *the eigenvalues of* A *are* $\{ (N-1), -1, -1, \dots, -1 \}$, where the *eigenvalue at* $-'1'$ *is of multiplicity* $'N-1''$.

  Similar result can be derived even when there are self loops at every vertex.

- Known Theorem: If G is a graph of diameter d, then the adjacency matrix of G has at least d+1 distinct eigenvalues (Thus, path graphs have distinct spectra).

- Graphs with few eigenvalues have been studied quite a bit [18].

Note: Interesting discussion on how non-isomorphic two graphs are is included in [10].
We now utilize Laplacian matrices of graphs to determine if they are isomorphic. This approach leads to **another algorithm for the problem which is more efficient**.

3.4 **Cholesky Decomposition: Another Algorithm for Graph Isomorphism**:

Let { G1,G2 } be diagonal matrices with vertex degrees of the two graphs. Also, let {A1,A2} be the adjacency matrices of those graphs. Hence, by definition, the Laplacian matrices of the graphs { L1,L2 } are given by

$$L_1 = G_1 - A_1 \quad and \quad L_2 = G_2 - A_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

It is well known that the Laplacian matrix of a graph is positive semi-definite. Thus, Cholesky Decomposition of Laplacian matrix exists (which is not necessarily unique). Such a decomposition can be computed efficiently. Thus, we have that

$$L_1 = N_1 N_1^T \quad and\ L_2 = N_2 N_2^T, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (4)$$

where N_1 *and* N_2 are lower triangular matrices.
If the graphs are isomorphic, we readily have that

$$L_2 = PLP^T = P\, N_1 N_1^T\, P^T = N_2 N_2^T \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots. (5)$$

Hence, it follows that

$$N_2 = PN_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Thus, a necessary and sufficient condition for the graphs to be isomorphic is that

$$N_2 N_1^{-1} = P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (7)$$

where P must be a Permutation matrix which is doubly stochastic and there is precisely *one* $p_{ij} = 1$ *in each row and each column.*

Hence, we have proved the following Lemma.

Lemma 6: Given the Cholesky Decomposition of Laplacian matrices of two graphs, they are isomorphic if and only if $N_2 N_1^{-1} = P$, where P is a Permutation matrix.

As in the case of algorithm in 3.3.2, the above test for graph isomorphism leads to another algorithm for graph isomorphism. This algorithm is known to be more efficient.
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4. Spectral Graph Theory: Interesting Proof of a Known Result:

Fact: While the adjacency matrix depends on the vertex labeling, its spectrum is a graph invariant.

We now provide an interesting proof of the above fact. In fact, the corollary of Lemma 7 is a much stronger result. We need the following well known theorem.

• **Rayleigh's Theorem**: The local optima of the quadratic form associated with a symmetric matrix A on the unit Euclidean hypersphere (i.e.{ X: $X^T X = 1$ }) occur at the eigenvectors with the corresponding value of the quadratic form being the eigenvalue.

Lemma 7. Eigenvalues of the adjacency matrix of an undirected graph, A are invariant under relabeling of the vertices.

Proof: By Rayleigh's theorm, eigenvalues of A are the local optimum of the associated quadratic form evaluated on the unit hypersphere. Thus, we need to reason that the quadratic form remains invariant under relabeling of the vertices. We have that

$$X^T A\, X \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} x_i x_j = \; x_1\left(x_{i_1} + x_{i_2} + \cdots x_{i_k}\right) + x_2\left(x_{j_1} + x_{j_2} + \cdots x_{j_l}\right) + \cdots + x_N\left(x_{N_1} + x_{N_2} + \cdots + x_{N_m}\right)$$

where, for instance, $\{\, i_1, i_2, \ldots .\, i_k\}$ are the vertices connected to the vertex 1 (one) (and similarly other vertices).

Now, from the above expression, it is clear that the quadratic form remains invariant under relabeling of the vertices. Specifically, relabeling just reorders the expressions.Thus, the eigenvalues of A remain invariant under relabeling of vertices **Q. E..D**

Corollary: Since the quadratic form remains invariant under relabeling of the vertices, the local optima of the quadratic form over various constraint sets remain invariant. For instance, the stable values (i.e. local optima of quadratic form associated with a symmetric matrix over the unit hypercube) remain same under relabeling of the vertices of graph.

Note: Consider a Homogeneous multi-variate polynomial associated with, say, a FULLY SYMMETRIC TENSOR. The local optima of such a homogenous form over various constraint sets such as Euclidean Unit Hypersphere, multi-dimensional hypercube remain invariant under relabeling of nodes of a non-planar graph. Effectively relabeling of vertices, reorders the monomials (terms in multivariate polynomial).

4.CONCLUSION:

In this research paper, results in spectral graph theory of structured graphs are discussed. Efficient algorithms for testing if two graphs are isomorphic are discussed.

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