Funayama's theorem revisited

Guram Bezhanishvili, David Gabelaia, and Mamuka Jibladze

[1] Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88003-8001, USA
gbezhani@math.nmsu.edu
[2] A. Razmadze Mathematical Institute
Tbilisi State University
University St. 2, Tbilisi 0186, Georgia
gabelaia@gmail.com
[3] A. Razmadze Mathematical Institute
Tbilisi State University
University St. 2, Tbilisi 0186, Georgia
mamuka.jibladze@gmail.com

1 Introduction

Let L be a lattice. We recall that L satisfies the *join infinite distributive law* (JID) if for each $a \in L$ and $S \subseteq L$, whenever $\bigvee S$ exists, then so does $\bigvee\{a \wedge s : s \in S\}$ and

$$a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}.$$

Similarly, L satisfies the *meet infinite distributive law* (MID) if whenever $\bigwedge S$ exists, then so does $\bigwedge\{a \vee s : s \in S\}$ and

$$a \vee \bigwedge S = \bigwedge\{a \vee s : s \in S\}.$$

Obviously each lattice that satisfies either (JID) or (MID) is distributive. A classic result in lattice theory is Funayama's theorem [5] stating that there is an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L iff L satisfies both (JID) and (MID).

Funayama's original proof was quite involved. For complete L, Grätzer [6, Sec. II.4] gave a more accessible proof of Funayama's theorem. The key ingredient of Grätzer's proof is to show that if L satisfies both (JID) and (MID), then the embedding of L into its free Boolean extension $B(L)$ is a complete lattice embedding. Then taking the MacNeille completion $\overline{B(L)}$ of $B(L)$ produces a complete Boolean algebra and the embedding $B(L) \hookrightarrow \overline{B(L)}$ preserves all existing joins and meets in $B(L)$. Thus, the composition $L \hookrightarrow B(L) \hookrightarrow \overline{B(L)}$ is a complete lattice embedding.

For complete L, Johnstone [8, Sec. II.2] gave a different proof of Funayama's theorem. Let L be a complete lattice satisfying (JID). Then L is a frame. Therefore, the poset $N(L)$ of all nuclei on L is also a frame, and the embedding $L \hookrightarrow N(L)$ is a frame homomorphism. Let $N(L)_{\neg\neg}$ be the Booleanization of $N(L)$; that is, the Boolean frame of regular nuclei on L. Thus, $N(L)_{\neg\neg}$ is a complete Boolean algebra and the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ is a frame embedding. If in addition L satisfies (MID), then the embedding $L \hookrightarrow N(L)_{\neg\neg}$ is a complete lattice embedding.

Our aim is to show that Grätzer's proof has an obvious generalization to the case when L is not necessarily complete, thus providing an accessible proof of Funayama's theorem in its full generality. If L is complete, we show that the complete Boolean algebras $\overline{B(L)}$ and $N(L)_{\neg\neg}$

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produced by Grätzer and Johnstone are isomorphic. This confirms a conjecture made by Leo Esakia in the early 1990s. We characterize lattices satisfying (JID) and (MID) by means of their Priestley spaces. Utilizing duality theory, we give alternative proofs of Funayama's theorem and of the isomorphism between $\overline{B(L)}$ and $N(L)_{\neg\neg}$. We also show that unlike Grätzer's proof, there is no obvious way to generalize Johnstone's proof to the non-complete case.

2 A generalization of Grätzer's proof

It is well known that the category BA of Boolean algebras is a reflective subcategory of the category DL of distributive lattices, and that the reflector sends each distributive lattice L to its free Boolean extension $B(L)$ which can be constructed as follows. Let X_L be the prime spectrum of L ordered by inclusion. For $a \in L$, let $\varphi(a) = \{x \in X_L : a \in x\}$. Then φ is a lattice embedding of L into the lattice of all up-sets of X_L. Let $B(\varphi[L])$ be the Boolean subalgebra of the powerset of X_L generated by $\varphi[L]$. The Boolean algebra $B(\varphi[L])$ is (isomorphic to) the free Boolean extension $B(L)$ of L.

Lemma 2.1. *Let L be a distributive lattice, let $B(L)$ be the free Boolean extension of L, and let $e\colon L \hookrightarrow B(L)$ be the canonical embedding.*

1. *If L satisfies* (JID)*, then e preserves all existing joins in L.*
2. *If L satisfies* (MID)*, then e preserves all existing meets in L.*

Let B be a Boolean algebra. We recall that the MacNeille completion of B is a complete Boolean algebra $\overline{B}$ such that there is a Boolean embedding $e\colon B \hookrightarrow \overline{B}$ that is join-dense in $\overline{B}$ (equivalently, e is meet-dense in $\overline{B}$). It is well known that the embedding e preserves all existing joins and meets in B.

Theorem 2.2 (Funayama's Theorem)**.** *Let L be a lattice.*

1. *L satisfies* (JID) *iff there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing joins in L.*
2. *L satisfies* (MID) *iff there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing meets in L.*
3. *L satisfies* (JID) *and* (MID) *iff there exists an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L.*

When L is complete, Theorem 2.2 yields Grätzer's proof of Funayama's theorem. Theorem 2.2 also has an obvious corollary for Heyting algebras.

Corollary 2.3. *Let L be a lattice.*

1. *If L is a Heyting algebra, then there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing joins in L.*
2. *If L is a co-Heyting algebra, then there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing meets in L.*
3. *If L is a bi-Heyting algebra, then there exists an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L.*

3 Nuclei, Booleanization, and Johnstone's proof

We recall that a *nucleus* on a meet-semilattice M is a map $j\colon M \to M$ satisfying

$$a \leq ja, \quad jja = ja, \quad j(a \wedge b) = ja \wedge jb,$$

for all $a, b \in M$. We will be mostly interested in nuclei on Heyting algebras and frames. If j is a nucleus on a frame L, then the set $L^j = \{a \in L : ja = a\}$ of its fixed points is a frame and $j\colon L \to L^j$ is an onto frame homomorphism whose right adjoint is the inclusion $L^j \hookrightarrow L$ [8, Sec. II.2].

For a frame L, let $N(L)$ be the set of all nuclei on L. If we order $N(L)$ pointwise, then it is well known that $N(L)$ is a frame and that $a \mapsto a \vee (-)$ is a frame embedding of L into $N(L)$ [8, Sec. II.2]. We call $j \in N(L)$ *regular* if $\neg\neg j = j$, where $\neg\neg$ is taken in $N(L)$. It is well known [8, Sec. II.2] that the set $N(L)_{\neg\neg}$ of all regular nuclei on L is a Boolean frame (complete Boolean algebra). Following the terminology of Banaschewski and Pultr [1], we call $N(L)_{\neg\neg}$ the *Booleanization* of $N(L)$.

Johnstone [8, Sec. II.2] proves that if L is a frame, then the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ is a frame embedding. In addition, the embedding $L \hookrightarrow N(L)$ preserves arbitrary meets iff L satisfies (MID). As $N(L)_{\neg\neg}$ is closed under arbitrary meets in $N(L)$, the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ preserves all meets in L iff L satisfies (MID). Therefore, the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ is a complete lattice embedding iff L satisfies both (JID) and (MID). This yields another proof of Funayama's theorem for complete lattices.

Theorem 3.1. *For each frame L, the complete Boolean algebras $\overline{B(L)}$ and $N(L)_{\neg\neg}$ are isomorphic.*

Thus, the Grätzer and Johnstone proofs of Funayama's theorem for complete L, although different, produce the same complete Boolean algebra in which there is a frame embedding of L. This confirms a conjecture made by Leo Esakia in the early 1990s.

4 Dual characterization of lattices satisfying (JID) and (MID) and Funayama's theorem

We assume the reader's familiarity with Priestley duality for distributive lattices [9, 10], and recall that the *Priestley space* of a distributive lattice L is the prime spectrum X_L of L ordered by inclusion. The topology on X_L is generated by the basis $\{\varphi(a) - \varphi(b) : a, b \in L\}$, where $\varphi(a) = \{x \in X_L : a \in x\}$ for each $a \in L$. Esakia duality for Heyting algebras [2] is a restricted version of Priestley duality. We recall that an *Esakia space* is a Priestley space X in which the down-set ${\downarrow}U$ of each clopen $U \subseteq X$ is clopen.

Dual spaces of co-Heyting algebras are Priestley spaces satisfying that U clopen implies ${\uparrow}U$ is clopen [3, 4]. We call such spaces *co-Esakia spaces*. Dual spaces of bi-Heyting algebras are Priestley spaces that are both Esakia spaces and co-Esakia spaces [3, 4]. We call such spaces *bi-Esakia space*.

Let X be a Priestley space and let $S \subseteq X$. As usual, $\overline{S}$ denotes the closure of S and int$S$ denotes the interior of S. Following [7], we let

$$\mathbf{J}S = X - {\downarrow}(X - \text{int}S) \text{ and } \mathbf{D}S = {\uparrow}\overline{S}.$$

It is easy to see that $\mathbf{J}S$ is the largest open up-set contained in S and $\mathbf{D}S$ is the smallest closed up-set containing S. Using this notation, being an Esakia space means that if U is an open up-set, then $\mathbf{D}U = \overline{U}$; and being a co-Esakia space means that if F is a closed up-set, then $\mathbf{J}F = \mathrm{int}F$.

Definition 4.1. *Let X be a Priestley space.*

1. *We call X a* J-space *if for each open up-set U, whenever $\mathbf{D}U$ is clopen, then $\mathbf{D}U = \overline{U}$.*
2. *We call X an* M-space *if for each closed up-set F, whenever $\mathbf{J}F$ is clopen, then $\mathbf{J}F = \mathrm{int}F$.*
3. *We call X a* JM-space *if X is both a J-space and an M-space.*

The condition defining a J-space weakens the condition defining an Esakia space. Similarly, the condition defining an M-space weakens the condition defining a co-Esakia space, and the condition defining a JM-space weakens the condition defining a bi-Esakia space.

Theorem 4.2. *Let L be a bounded distributive lattice and let X_L be the Priestley space of L.*

1. *L satisfies* (JID) *iff X_L is a J-space.*
2. *L satisfies* (MID) *iff X_L is an M-space.*
3. *L satisfies* (JID) *and* (MID) *iff X_L is a JM-space.*

Theorem 4.2 can be used to give an alternative proof of the nontrivial implication in Funayama's theorem by means of Priestley duality. Indeed, let L satisfy (JID) and let X_L be the Priestley space of L. By Theorem 4.2, X_L is a J-space. Let $S \subseteq L$ be such that $\bigvee_L S$ exists. We let U be the open up-set $\bigcup\{\varphi(s) : s \in S\}$. Then $\varphi(\bigvee_L S) = \mathbf{D}U$, so $\mathbf{D}U$ is clopen, and as X_L is a J-space, $\mathbf{D}U = \overline{U}$. Therefore, $\overline{U}$ is clopen. Since $B(L)$ is isomorphic to the Boolean algebra of clopen subsets of X_L, as $\overline{U}$ is clopen, the join of the image of S in $B(L)$ exists and is equal to $\overline{U}$. Thus, the canonical embedding $\varphi\colon L \hookrightarrow B(L)$ preserves all existing joins in L. A similar argument gives that if L satisfies (MID), then X_L is an M-space, and so φ preserves all existing meets in L. Thus, if L satisfies both (JID) and (MID), then X_L is a JM-space, and so φ preserves all existing joins and meets in L. Taking the MacNeille completion of $B(L)$ then completes the proof.

Theorem 4.2 can also be used to obtain an alternative proof of Theorem 3.1, as well as to show that unlike Grätzer's proof, Johnstone's proof has no obvious generalization to the non-complete case. The details as well as all the missing proofs will be published in the full version of the paper, which will appear in Algebra Universalis.

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