

Two-layer modal logics: from fuzzy logics to a general framework

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Mathematical Fuzzy Logic (MFL) started as the study of logics based on left-continuous t-norms,¹ most prominently Łukasiewicz logic \mathbb{L} , Gödel–Dummett logic \mathbb{G} , Product logic \mathbb{P} , Hájek logic \mathbb{BL} , and the system \mathbb{MTL} introduced by Esteva and Godo. In the last years, the scope of MFL has been progressively expanded by considering weaker logical systems characterized by their completeness with respect to a semantics of linearly algebras (such as chains endowed with a uninorm, or a non-commutative t-norm to interpret $\&$, or other kinds of linearly ordered residuated lattices).

There have been some proposals for a general framework to deal in a uniform way with this growing family of logics. The first one appeared in [12] when Hájek and Cintula introduced that classes of core and Δ -core fuzzy logics, understood as axiomatic expansions of \mathbb{MTL} and \mathbb{MTL}_Δ , resp. A wider framework, encompassing weaker systems and stronger expansions, has been proposed in [3] based on the notion of weakly implicative semilinear logic. In both approaches MFL retains what we can arguably see as its defining feature, namely the study of systems of non-classical logics with a semantics based on (linearly ordered) scales of degrees of truth. This is what makes these logics specially suited for the study of gradual aspects of vagueness and imprecision, found in sentences like ‘it is heavily raining’ or ‘that man is tall’.

A conceptually different issue, that of uncertainty, has also been addressed inside MFL. The main idea, introduced in [13] and later developed by Hájek in his monograph [11], is that one could use probability to determine the truth degree of statements such as ‘tomorrow it will probably rain’ or ‘the probability that tomorrow it will rain is high’. Indeed, one takes classical logic and its formulae φ to describe crisp events, introduces a new modal operator P which can be applied on them to create atomic modal formulae $P\varphi$ which may be read as ‘probably φ ’ (or better ‘the probability of φ is high’), and finally these atomic modal formulae are combined by using the connectives of Łukasiewicz logic. What we obtain is a *two-layer modal fuzzy logic* built on atomic formulae $P\varphi$ whose truth values are given by a probability measure. Several works have followed this idea with variations. In [9] Godo, Esteva and Hájek replaced Łukasiewicz logic on the second layer by $\mathbb{L}\mathbb{I}$, but kept classical logic for non-modal formulae. The logic $\mathbb{L}\mathbb{I}$, with its expanded language, enabled them to deal with conditional probability. Flaminio and Montagna also considered conditional probability in [7], and Godo and Marchioni investigated coherent conditional probabilities in [10]. Marchioni also proposed a class of *logics of uncertainty* in [14] with different kinds of measures (besides probability) to quantify the uncertainty of events. In all of these works classical logic has been kept as the underlying logic for non-modal formulae.

However, if one wants to deal with uncertainty and vagueness at once, i.e. with the probability of vague events, as in ‘tomorrow it will probably rain *heavily*’, the two-layer paradigm can still be useful provided that the underlying classical logic is substituted by a fuzzy logic. This idea has been also investigated in some works, as [5] where finite Łukasiewicz systems \mathbb{L}_n are taken as the logics of vague events. Other recent works along these lines are surveyed in [6]. There has even been a first attempt at an abstract theory of two-layer modal fuzzy logics in [15]; but it is restricted to the family of core fuzzy logics which, as argued above, has become too narrow to contain the current scope of MFL.

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¹For a more detailed historical account see [1].

The primary aim of this paper is to provide a new general framework for two-layer modal fuzzy logics that encompasses the current state of the art and paves the way for future development. Actually, we will obtain much more than this. Indeed, we show how one can construct a modal logic (for an arbitrary modality, not necessarily read as a probability) over an *arbitrary non-classical logic* (under certain technical requirements). Therefore, we need not assume that the starting logic is fuzzy, and we can develop a general theory of two-layer modal logics, showing how the methods used in the fuzzy literature can lead to completeness results using very few properties of the underlying logics. As a semantics, we propose particular kinds of *measured Kripke Frames* and prove corresponding completeness theorems. As an illustration we will obtain Hájek's completeness result for Fuzzy Probability logic $\text{FP}(\mathbb{L})$ over Łukasiewicz logic as a particular corollary of our general approach.

1 The framework

This section presents the basic definitions and notational conventions for the paper (for further information on Algebraic Logic notions see [8, 4]). The definitions of a propositional language \mathcal{L} , the free term algebra $\mathbf{Fm}_{\mathcal{L}}$ over a denumerable set of generators (propositional variables), and finitary Hilbert-style proof systems are as usual. Let us introduce the notion of propositional logic that we use in this paper.

Convention 1. *Let \mathcal{L} be a language containing at least a truth constant $\bar{1}$ and binary connectives \rightarrow and \vee . In this paper a propositional logic \mathbb{L} in \mathcal{L} is a finitary lattice-disjunctive weakly implicative logic (as studied in [3]). In more details, this means that \mathbb{L} is identified with the provability relation $\vdash_{\mathbb{L}}$ on $\mathbf{Fm}_{\mathcal{L}}$ given by a finitary Hilbert-style system such that:²*

$$\begin{aligned} \vdash_{\mathbb{L}} \varphi \rightarrow \varphi \quad \varphi, \varphi \rightarrow \psi \vdash_{\mathbb{L}} \psi \quad \varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_{\mathbb{L}} \varphi \rightarrow \chi \quad \varphi \dashv\vdash_{\mathbb{L}} \bar{1} \rightarrow \varphi \\ \varphi \leftrightarrow \psi \vdash_{\mathbb{L}} \circ(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \leftrightarrow \circ(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n) \quad \text{for every } n\text{-ary } \circ \in \mathcal{L} \text{ and } i < n. \\ \vdash_{\mathbb{L}} \varphi \rightarrow \varphi \vee \psi \quad \vdash_{\mathbb{L}} \psi \rightarrow \varphi \vee \psi \quad \varphi \rightarrow \chi, \psi \rightarrow \chi \vdash_{\mathbb{L}} \varphi \vee \psi \rightarrow \chi \\ \Gamma, \varphi \vdash_{\mathbb{L}} \chi \quad \text{and} \quad \Gamma, \psi \vdash_{\mathbb{L}} \chi \quad \text{imply} \quad \Gamma, \varphi \vee \psi \vdash_{\mathbb{L}} \chi \end{aligned}$$

We recall now the basics of semantics. Note that our logics are algebraically implicative with a truth definition given by the single equation $x \vee \bar{1} \approx \bar{1}$. Let us fix a logic \mathbb{L} in a language \mathcal{L} ; then \mathcal{L} -algebras are algebras with signature \mathcal{L} and homomorphisms from $\mathbf{Fm}_{\mathcal{L}}$ to an \mathcal{L} -algebra A are called *A-evaluations*. For an \mathcal{L} -algebra A we define the set $F_A = \{x \mid x \vee^A \bar{1}^A = x\}$.

Definition 2. *We say that A is an \mathbb{L} -algebra, $A \in \mathbb{L}$ in symbols, if*

- *for each $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathbb{L}} \varphi$, we have that for each A -evaluation e , if $e[\Gamma] \subseteq F_A$, then $e(\varphi) \in F_A$,*
- *for each $x, y \in A$, if $\{x \rightarrow^A y, y \rightarrow^A x\} \subseteq F_A$, then $x = y$.*

\mathbb{L} is in fact a quasivariety and it is the equivalent algebraic semantics of \mathbb{L} in the sense of [2]. A non-trivial \mathbb{L} -algebra A is (finitely) *subdirectly irreducible relative to \mathbb{L}* if for every (finite non-empty) subdirect representation α of A with a family $\{A_i \mid i \in I\} \subseteq \mathbb{L}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. $\mathbb{L}_{\text{R(F)SI}}$ denotes the class of all (finitely) subdirectly irreducible algebras relative to \mathbb{L} . Of course $\mathbb{L}_{\text{RSI}} \subseteq \mathbb{L}_{\text{RFSI}}$.

Definition 3. *Let \mathbb{L} be a logic and $\mathbb{K} \subseteq \mathbb{L}_{\text{RFSI}}$. We say that \mathbb{L} has (finite) strong \mathbb{K} -completeness, SKC (or FSKC resp.) whenever for each (finite) theory $\Gamma \cup \{\varphi\}$ holds that $\Gamma \vdash_{\mathbb{L}} \varphi$ iff for each $A \in \mathbb{K}$ and each A -evaluation e we have $e(\varphi) \in F_A$ whenever $e[\Gamma] \subseteq F_A$.*

²We write ' $\varphi \leftrightarrow \psi$ ' for ' $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi\}$ ', ' $T \vdash S$ ' for ' $T \vdash \varphi$ for each $\varphi \in S$ ', and ' $T \dashv\vdash S$ ' for ' $T \vdash S$ and $S \vdash T$ '.

Algebraically we can say that \mathbb{L} has FS \mathbb{K} C (or SKC resp.) if \mathbb{K} generates \mathbb{L} as a (σ) -quasivariety. Note that every logic has SL $_{\text{RFSI}}$ C (and hence SL $_{\text{RFSI}}$ C). If \mathbb{L} is a fuzzy logic, then \mathbb{L}_{RFSI} is the class of \mathbb{L} -chains.

2 The core theory

Let us fix two logics \mathbb{L}_1 and \mathbb{L}_2 in disjoint languages \mathcal{L}_1 and \mathcal{L}_2 such that $\Box \notin \mathcal{L}_1 \cup \mathcal{L}_2$. Further we fix two classes of algebras $\mathbb{K}_i \subseteq (\mathbb{L}_i)_{\text{RFSI}}$, $i \in \{1, 2\}$. We define three kinds of formulae of a two-level language $Fm_{\mathcal{L}_2(\mathcal{L}_1)}^{\text{Var}}$ over the set of variables Var :

- non-modal formulae from $Fm_{\mathcal{L}_1}^{\text{Var}}$,
- atomic modal formulae of the form $\Box\varphi$, for $\varphi \in Fm_{\mathcal{L}_1}^{\text{Var}}$,
- modal formulae resulting from atomic ones by connectives from \mathcal{L}_2 .

Definition 4. The minimal \mathbb{L}_2 -modal logic over logic \mathbb{L}_1 (denoted by $\mathbb{L}_2(\mathbb{L}_1)$) has formulae $Fm_{\mathcal{L}_2(\mathcal{L}_1)}^{\text{Var}}$ and an axiomatic system consisting of

- the axioms and rules of \mathbb{L}_1 for non-modal formulae,
- axioms and rules of \mathbb{L}_2 for modal formulae,
- and the following congruence rule for each pair of non-modal formulae φ and ψ :

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi \quad (\text{CONGR})$$

An n -ary modal rule has n non-modal premises and a modal conclusion. An \mathbb{L}_2 -modal logic over a logic \mathbb{L}_1 is an extension of $\mathbb{L}_2(\mathbb{L}_1)$ by some modal rules.

We understand rules as schemata, i.e., for each substitution σ on $Fm_{\mathcal{L}_1}^{\text{Var}}$, if $\varphi_1, \dots, \varphi_n \vdash \Psi$ is a modal rule then $\sigma\varphi_1, \dots, \sigma\varphi_n \vdash \sigma\Psi$ is also a modal rule. We define the notion of proof in a modal logic in the usual way. One can imagine that the proof consists of three separate parts: proving non-modal formulae, application of the modal rules on proved non-modal formulae, and proving modal formulae.

Definition 5. A \mathbb{K}_1 -based \mathbb{K}_2 -measured Kripke frame is a system $\mathbf{F} = \langle W, (A_w)_{w \in W}, \mathbf{B}, \mu \rangle$ where W is a set (of possible worlds), $A_w \in \mathbb{K}_1$ for each $w \in W$, $\mathbf{B} \in \mathbb{K}_2$ and μ is a partial mapping $\mu: \prod_{w \in W} A_w \rightarrow B$.

Note the difference from the ‘traditional’ approach: in order to prove the completeness theorems in the full generality we cannot assume that all A_w s are the same; we call such frames *uniform* and we will see later in which cases we can restrict ourselves to such frames.

Definition 6. A Kripke model \mathbf{M} over \mathbb{K}_1 -based \mathbb{K}_2 -measured Kripke frame $\mathbf{F} = \langle W, (A_w)_{w \in W}, \mathbf{B}, \mu \rangle$ is a tuple $\mathbf{M} = \langle \mathbf{F}, (e_w)_{w \in W} \rangle$ where:

- $e_w: Fm_{\mathcal{L}_1}^{\text{Var}} \rightarrow A_w$ is an A_w -evaluation,
- for each non-modal formula φ , the element $\varphi_{\mathbf{M}} \in \prod_{w \in W} A_w$ defined as $\varphi_{\mathbf{M}}(w) = e_w(\varphi)$ belongs to the domain of μ .

The truth value of atomic modal formulae is defined (uniformly for all worlds) as:

$$\|\Box\varphi\|_{\mathbf{M}} = \mu(\varphi_{\mathbf{M}});$$

and the truth value of non-atomic modal formulae is (uniformly) computed by using operations from \mathbf{B} .

We say that \mathbf{M} satisfies the (non-)modal formula Ψ (ψ resp.) whenever $\|\Psi\|_{\mathbf{M}} \in F_B$ (or $e_w[\psi] \in F_{A_w}$ for each $w \in W$ respectively).

Finally we say that \mathbf{F} is a frame for an L_2 -modal logic over a logic L_1 if all its additional modal rules are valid in all Kripke models over \mathbf{F} , i.e. the conclusion of a modal rule is satisfied in all models of over \mathbf{F} which satisfy all its premises.

Next we state the main theorem, the completeness of an L_2 -modal logic over a logic L_1 . We will see that the form/strength of the completeness we obtain depends on the form/strength of the completeness of the logics L_2 and L_1 . The proof has two main ingredients: Hájek's idea from [11] of a translation of formulae and proofs from an L_2 -modal logic over a logic L_1 into the logic L_2 and the authors' characterization of completeness properties from [3].

Theorem 7. *Let L be an L_2 -modal logic over a logic L_1 such that L_i has $S\mathbb{K}_iC$. Then the following are equivalent for each non-modal theory T , modal theory Γ , and a modal formula Φ :*

- $\Gamma, T \vdash_L \Phi$
- for each \mathbb{K}_1 -based \mathbb{K}_2 -measured Kripke frame \mathbf{F} for L and each Kripke model \mathbf{M} over \mathbf{F} holds that \mathbf{M} satisfies Φ whenever it satisfies all formulae from Γ and T .

The same equivalence holds if L_2 has $FS\mathbb{K}_2C$ only but at the price of restricting to finite Γ and T and additional assumptions that L_1 is a locally finite and L has only finitely many additional modal rules.

Note that any L_2 -modal logic over a logic L_1 enjoys completeness w.r.t. its $(L_1)_{\text{RFSI}}$ -based $(L_2)_{\text{RFSI}}$ -measured Kripke frames and if L_1 enjoys completeness w.r.t. a single algebra, then we can restrict ourselves to uniform frames/models.

3 A case study: fuzzy probability logic over Boolean logic

As a case study we repeat the definition of *Fuzzy Probability logic over Łukasiewicz logic* \mathbb{L} [13, 11], denoted as $FP(\mathbb{L})$ for short, and show how its completeness follows from our main theorem. Recall that $FP(\mathbb{L})$ is given by the following axiomatic system:

- axioms of classical propositional logic Bool for non-modal formulae and axioms \mathbb{L} for modal ones,
- *modus ponens* rules for both non-modal and modal formulae,
- axioms $\Box\varphi \rightarrow (\Box(\varphi \rightarrow \psi) \rightarrow \Box\psi)$, $\Box(\neg\varphi) \leftrightarrow \neg\Box(\varphi)$, and $\Box(\varphi \vee \psi) \leftrightarrow [(\Box\varphi \rightarrow \Box(\varphi \wedge \psi)) \rightarrow \Box\psi]$,
- modal rule $\varphi \vdash \Box\varphi$.

Clearly the logic $FP(\mathbb{L})$ is an \mathbb{L} -modal logic over Bool with only finitely many additional modal rules; Bool is locally finite and enjoys $S2C$; and \mathbb{L} enjoys $FS[0, 1]_{\mathbb{L}}C$. Thus we can use our main theorem. In order to obtain exactly Hájek's formulation we also observe that $\mathbf{F} = \langle W, (A_w)_{w \in W}, \mathbf{B}, \mu \rangle$ is a $\{2\}$ -based $\{[0, 1]_{\mathbb{L}}\}$ -measured Kripke frame for $FP(\mathbb{L})$ iff μ is finitely additive probability measure.

Theorem 8 ([11, Theorem 8.1.14]). *Let T and Γ be finite non-modal and modal (resp.) theories in $FP(\mathbb{L})$ and let Φ be a modal formula. Then the following are equivalent:*

- $\Gamma, T \vdash_{FP(\mathbb{L})} \Phi$
- $\|\Phi\|_{\mathbf{M}} = 1$ for all Kripke models \mathbf{M} , satisfying Γ and T , over all $\{2\}$ -based $[0, 1]_{\mathbb{L}}$ -measured Kripke frames, where μ is a finitely additive probability measure.

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