Canonical formulas via locally finite reducts and generalized dualities

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Axiomatizability, the finite model property (FMP), and decidability are some of the most frequently studied properties of non-classical logics. One of the first general methods of axiomatizing large classes of superintuitionistic logics (si-logics for short) was developed by Jankov [8]. For each finite subdirectly irreducible Heyting algebra A, Jankov designed a formula that encodes the structure of A. The main property of the Jankov formula $\chi(A)$ is that a Heyting algebra B refutes $\chi(A)$ iff A is isomorphic to a subalgebra of a homomorphic image of B. In [9] Jankov utilized this method to show that there are continuum many si-logics; in fact, continuum many si-logics axiomatized by Jankov formulas. However, not every si-logic is axiomatizable by Jankov formulas.

Model-theoretic analogues of Jankov formulas were developed by de Jongh [10] for si-logics and by Fine [6] for modal logics. In [7] Fine introduced the concept of a subframe logic, axiomatized all transitive subframe logics by means of subframe formulas, and proved that each transitive subframe logic has the FMP. Zakharyaschev generalized Fine's approach, developed the model-theoretic theory of canonical formulas (in [12] for si-logics and in [11, 13] for modal logics), and showed that each si-logic and each transitive modal logic is axiomatizable by canonical formulas. See [5, Ch. 9] for an overview of these results.

In this talk, which is based on joint work with G. Bezhanishvili [1, 2, 3, 4], I will discuss an algebraic approach to the method of canonical formulas. I will mostly concentrate on the case of si-logics. But I will also review the case of modal logics and possible generalizations to substructural logics.

For si-logics the method boils down to identifying appropriate locally finite reducts of Heyting algebras. The variety of Heyting algebras has two well-behaved locally finite reducts, the variety of bounded distributive lattices and the variety of implicative semilattices. The variety of bounded distributive lattices is generated by the $\to$-free reducts of Heyting algebras, while the variety of implicative semilattices by the $\vee$-free reducts. Each of these reducts gives rise to canonical formulas that generalize Jankov formulas and provide an axiomatization of all si-logics.

For a finite subdirectly irreducible Heyting algebra A and $D \subseteq A^2$, we design the $(\wedge, \to)$-canonical formula of A that encodes fully the structure of the $\vee$-free reduct of A, and only partially the behavior of $\vee$. We also design the $(\wedge, \vee)$-canonical formula of A that encodes fully the structure of the $\to$-free reduct of A, and only partially the behavior of $\to$. We prove that every si-logic is axiomatizable by $(\wedge, \to)$-canonical formulas as well as by $(\wedge, \vee)$-canonical formulas. We discuss the similarities and differences between these two kinds of formulas. Via the generalized Esakia duality of Heyting algebras and $(\wedge, \to)$-homomorphisms, we show that $(\wedge, \to)$-canonical formulas are algebraic analogues of Zakharyaschev's canonical formulas.

One of the main ingredients of our formulas is a designated subset D of pairs of elements of a finite subdirectly irreducible Heyting algebra A. The obvious two extreme cases are when

$D = \emptyset$ or $D = A^2$. When $D = A^2$, we show that the $(\wedge, \to)$ and $(\wedge, \vee)$-canonical formulas of A are equivalent to the Jankov formula of A. On the other hand, when $D = \emptyset$, the $(\wedge, \to)$-canonical formulas produce the algebraic counterpart of subframe formulas, which axiomatize all subframe si-logics. In the $(\wedge, \vee)$-case, $D = \emptyset$ produces a new class of si-logics, which we term stable si-logics. As in the case of subframe logics, we prove that all stable si-logics have the FMP. We show that there are continuum many stable si-logics, and give examples showing that the classes of stable, subframe, and join-splitting si-logics are incomparable.

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